Let us start from the ODE in $\mathbb{R}^d$

(*) $\dot{X}(t) = b(X(t)), \quad X(0) = x \in \mathbb{R}$

for some $b : \mathbb{R}^d \to \mathbb{R}^d$ in $C^1_b(\mathbb{R}^d)$.

Now, we consider the following random perturbation of (*):

(**) $dX(t) = b(X(t))dt + \sqrt{2\varepsilon} g(X(t))dW(t)$

where $\varepsilon > 0$ is a small parameter

$\sigma : \mathbb{R}^d \to C(\mathbb{R}^d)$ of class $C^1_b$

and $W(t)$ is a $d$-dimensional Brownian motion

You can think of

$\mathcal{S}_\varepsilon = C_0([0, +\infty); \mathbb{R}^d)$

$\mathcal{B}(C_0([0, +\infty); \mathbb{R}^d)) = \mathcal{F}$

$\mathcal{F}$ Wiener-measure

and $W(t)(\omega) = \omega(t), \quad \forall \omega \in \mathcal{S}_\varepsilon, \quad \forall t \geq 0$.

We denote by $X_\varepsilon(t)$ the solution of (**)

and $\varepsilon > 0$ we denote by $X_\varepsilon(t)$ the solution of (**).
We have the following fact

**Prop.** Let $T > 0$ and $x \in \mathbb{R}^d$.

\[
E \left( \sup_{t \in [0,T]} \left| X^x_0(t) - X^x_T(t) \right|^2 \right) \leq C T e^{-\lambda T} \quad \text{for any } \lambda > 0.
\]

In particular,

\[
\lim_{\lambda \to 0} E \left( \sup_{t \in [0,T]} \left| X^x_0(t) - X^x_T(t) \right|^2 \right) = 0.
\]

This means that

\[
X^x_0 \to X^x_T \quad \text{in } L^2(\Omega; C([0,T]; \mathbb{R}^d)).
\]

In particular, if we denote by

\[
\nu^x_T = L(X^x_T) \quad \text{in } C_{x,T} = \{ \psi : [0,T] \to \mathbb{R}^d \quad \text{with } \psi(0) = x \},
\]

we have that

\[
\nu^x_T \to \delta_{x^0}
\]

our purpose is to study large deviations of $\nu^x_T$ from its limit $\delta_{x^0}$ but we have to give a meaning to this.
Let \( E \) be a Polish Space (complete separable metric space) (think of a Banach Space).

Suppose \( \mu \in \mathcal{P}(E) \) such that

\[
\mu \longrightarrow \delta_p, \quad \text{for some } p \in E.
\]

Then

\[
\forall U \ni p \quad \mu(U) \rightarrow 0.
\]

This can be said that

as \( \varepsilon \downarrow 0 \) \( \mu \) sees \( p \) as being typical.

In other words, if

\[
T \in \mathcal{B}(p, \varepsilon), \quad \text{then } T \text{ describes a "deviant" behavior.}
\]

It is of interest to see/determine how deviant a particular event \( T \) is. That is, if

\[
T \in \mathcal{B}(p) \quad \text{with } p \notin T,
\]

one is interested in the rate \( \mu(T) \) is going to zero.

If one is interested in event which are "very deviant" in the sense that

\( \mu(T) \) goes to zero exponentially fast,
then one is studying
the LARGE DEVIATIONS of the
family

(SIMPLE EXAMPLE)

Let $\xi_1, \xi_2, \ldots$ be a i.i.d sequence
on $(\Omega, \mathcal{F}, P)$

Let $E\xi_i = \mu \in \mathbb{R}$ \hspace{1cm} \text{Var}\xi_i = 6^2 e(0, +\infty)$

and let $S_n = \xi_1 + \ldots + \xi_n$.

(I) STRONG LAW
OF LARGE NUMBERS says:

$\frac{S_n}{n} \to \mu \quad \text{P-a.s.}$

Hence, if we define

$P_n := P\left(\frac{S_n}{n}\right) \quad P_n \to \delta_\mu$

(II) CENTRAL LIMIT
THEOREM says:

$\frac{S_n - \mu}{\sqrt{n}} \to N(0, 1) \quad \text{as} \ n \to \infty$

Hence

$\forall a > 0 \quad \text{P}\left(\frac{S_n - \mu}{\sqrt{n}} > \frac{a}{\sqrt{n}}\right) \to 1 - \Phi\left(\frac{a}{\sqrt{6}}\right) = 0$

These are the so-called
NORMAL DEVIATIONS
(III) Now let us estimate
\[ P \left( \frac{S_n - \mu}{n^{a/2}} > \frac{a}{\sqrt{n}} \right) \text{ with } a < \frac{1}{2} \]
these are larger denominations.

By the Chebychev inequality
\[ P \left( \frac{|S_n - \mu|}{n^{a/2}} > \frac{a}{\sqrt{n}} \right) \leq \frac{n^{2a}}{a^2} \frac{V(S_n)}{(n-2a)^2} \]
\[ = (\frac{a}{\sqrt{n}})^2 \frac{1}{n^{1-2a}} \to 0 \]

What is possible to prove is that
if \( a = 0 \), we have
\[ P \left( \frac{S_n - \mu}{n} > a \right) \approx e^{-nI(a)} \text{, } I(a) = 0 \]
where
\[ I : [0, \infty) \to [0, \infty) \]
\[ I(a) \geq 0 \text{ for } a \geq 0. \]
\[ I(a) = 0 \iff a = 0. \]

By symmetry
\[ P \left( \frac{S_n - \mu}{n} < a \right) \approx e^{-nI(a)} \text{ for } a < 0 \]
Hence we have
\[ I : \mathbb{R} \to [0, \infty] \]
with \( I(a) \geq 0 \) for \( a \in \mathbb{R} \)
and \( I(a) = 0 \iff a = 0 \).

\[ I(a) = \begin{cases} 
0 \quad &a = 0 \\
\frac{1}{2} \log 2 + \frac{1}{2} \log(2 + \frac{1}{a}) + \frac{1}{2} \log(2 + \frac{1}{a}) + 2 \log \left( \frac{1}{2} \right) \quad &a < 0 \\
0 \quad &\text{otherwise} \end{cases} \]
DEFINITION
of LARGE DEVIATIONS

Let \( E \) be a Polish space.

**Def. 1**
A function \( I: E \to [0, \infty] \) is called ACTION or RATE FUNCTIONAL if

1. \( I \) \( \geq \) \( \infty \)
2. \( I \) is lower semi-continuous (i.e.
   \[ I(x) \leq C \quad \text{if } C \in E \text{ is closed}, \quad \forall C \in E \]
3. \( I \) has compact level sets.

**Def. 2**
Let \( \{P_\varepsilon\}_\varepsilon \) be a family of probability measures on \((E, \mathscr{B} (E))\).

\( \{P_\varepsilon\}_\varepsilon \) satisfies a LARGE DEVIATION PRINCIPLE (LDP) WITH RATE \( \varepsilon \) AND RATE FUNCTIONAL \( I \) if:

a) \( \limsup_{\varepsilon \to 0} \varepsilon \log P_\varepsilon (C) \leq - \inf_{x \in C} I(x) \quad \forall C \in E \text{ closed} \)

b) \( \liminf_{\varepsilon \to 0} \varepsilon \log P_\varepsilon (A) \geq - \inf_{x \in \overline{A}} I(x) \quad \forall A \in E \text{ open} \)

REM. The action functional is unique.
If \( \inf_{x \in A} I(x) = \inf_{x \in A} I(x) \),
then
\[
\gamma_{2}(E) \leq 3 \inf_{x \in E} I(x)
\]

**COMMENTS**

1) Why to distinguish between open and closed sets?
   One might try to use
   \[
   \lim_{z \to 0} z \log \gamma_{2}(E) = - \inf_{x \in E} I(x)
   \]
   \( \forall E \in \mathcal{B}(E) \).

   This would be too restrictive:
   For example, let \( \gamma_{2}(\{x\}) = 0 \), \( \forall x \in E \).
   Then for \( S = \{x\} \) we would have
   
   \[
   x = \inf_{x \in \{x\}} I(x) = I(x) \,
   \]
   \( \forall x \in S \). Hence we would have \( I \equiv \infty \).

2) If \( I \) is continuous
   then, for any \( A \in \mathcal{E} \) open
   \[
   \inf_{x \in A} I(x) = \inf_{x \in A} I(x)
   \]
3) The role played by open and closed sets is the same as their role in weak convergence of measures. 
\[ \mathcal{E} \rightarrow \mathcal{E} \] iff.
1) \( \limsup_{n \to \infty} \mathcal{E}_n(\mathcal{C}) \leq \mathcal{E}(\mathcal{C}) \) \( \forall \mathcal{C} \) closed.
2) \( \liminf_{n \to \infty} \mathcal{E}_n(\mathcal{A}) \geq \mathcal{E}(\mathcal{A}) \) \( \forall \mathcal{A} \) open.

4) In the definition of action functional we asked both \( \{ \mathcal{I}_{\mathcal{C}} \} \) closed and \( \{ \mathcal{I}_{\mathcal{C}} \} \) compact and this is clearly plausible.

The role of \( (\dagger) \) is to guarantee that \( \mathcal{E}_\mathcal{E} \) is exponentially tight, that is:

\[ \forall \mathcal{M} \in \mathcal{K}_{\mathcal{M}} \text{ compact} \]
\[ \limsup_{n \to \infty} \mathcal{E}(\log \mathcal{E}_n(K_{\mathcal{M}})) \leq -M \]
\[ \iff \mathcal{E}(K_{\mathcal{M}}) \leq e^{-\frac{1}{3}(M/3)} \]
\[ \forall \mathcal{E} > 0 \exists \mathcal{E} > 0 : \forall \mathcal{M} \in \mathcal{K}_{\mathcal{M}} \text{ compact} \]
From the IUP it follows that \( \inf_{x \in E} I(x) = 0 \).

Actually, as \( E \) is closed we have \( \limsup_{\epsilon \to 0} \epsilon \log \frac{1}{\nu(E)} \leq - \inf_{x \in E} I(x) \).

Moreover, if \( I \) is compact and as \( I \) is lower semi-continuous, \( I \) admits a minimum on any compact.

This means that \( \exists \bar{x} \in E \text{ s.t. } I(\bar{x}) = 0 \).

In most of the examples, \( \bar{x} \) is unique and coincides with what we get from the SLLN.

The Freidlin-Wentzell formulation:

\[
\limsup_{\epsilon \to 0} \epsilon \log \frac{1}{\nu(E)} \leq - \inf_{x \in E} I(x), \quad \forall C \in \mathcal{C}, \quad \text{closed}
\]

\( \forall \epsilon_0 > 0, \delta > 0, \exists \epsilon(0, \epsilon_0) \quad \exists \delta \in (0, \epsilon_0) \text{ s.t. } \nu(E) \leq \exp\left(-\frac{1}{\epsilon} (r_0 - \delta)\right) \quad \forall \epsilon > 0 \)
here \( I^r = \{ I \leq r \} \) is the level set

\[
\liminf_{t \to 0} \frac{3}{2} \Psi^2 (A) \geq - \inf_{x \in A} I(x), \quad \text{if} \quad A \subset E \quad \text{open}
\]

\[
\forall x_0 \in E, \quad \delta > 0, \quad \varepsilon > 0 \quad \exists \varepsilon_0 > 0 \quad \text{s.t.}
\]

\[
\Psi^2 (B(x_0, \delta)) \geq \exp \left( - \frac{1}{2} (I(x_0) + \varepsilon) \right), \quad \forall \varepsilon < \varepsilon_0.
\]

Hence, let us go back to our original problem

\[
dX (t) = b(X (t)) \, dt + \sqrt{\varepsilon} \, dW (t)
\]

(\text{here I take} \( \varepsilon = 1 \))

we denote

\[
\Psi^2 \equiv \mathcal{L} (X^2) \quad \text{in} \quad \mathcal{C} ([0, T]; \mathbb{R}^d), \quad \text{where} \quad T > 0 \quad \text{is fixed}.
\]

we have seen that \( \Psi^2 \to \delta_{x_0} \), as \( \varepsilon \to 0 \).

we want to show that

\[
\{ \Psi^2 \} \quad \text{satisfies a LDP with respect to a suitable action functional with rate} \quad q > 0.
\]
we first consider the case \( b = 0, \, x = 0 \)

In this case \( P_e = \mathcal{L}(\text{VEW}) \) and \( W(t) \) is the Standard Brownian motion.

**Theorem**

The family \( \{M_{\phi} = \{\mathcal{L}(\text{VEW})\} \} \) satisfies a LDP with rate \( 2 \) and action functional \( I(\phi) = \frac{1}{2} \int_0^T |\phi'(t)|^2 \, dt \), if \( \phi \in \mathcal{W}^2(\mathbb{R}^1) \) with \( \phi(0) = 0 \).

First of all, is \( I \) a good action functional?

1) Clearly, \( I \neq \infty \).
2) Is \( I \) compact?

Yes, this follows from the Ascoli-Arzela theorem, as the function in \( \{I_{\phi} \} \) are equicontinuous and equibounded.

Now, in order to give a proof of the theorem, we use the Freidlin-Wentzell formulation.
Let $\varphi \in \mathcal{E}$, $\delta > 0$, $\varepsilon > 0$

we want to find $\varepsilon_0 > 0$

such that

$$M_2(B(\varphi, \delta)) = \mathbb{P}\left( \|\mathbb{W}(1) - \varphi\|_E < \delta \right) \geq \exp\left( -\frac{1}{3} (\mathbb{I}(\varphi) + \varepsilon) \right),$$

$$\varepsilon_0 > 0.$$ 

**Proof:**

Now

$$\mathbb{P}\left( \|\mathbb{W}(1) - \varphi\|_E < \delta \right) = \mathbb{P}\left( \|\mathbb{W} - \varphi\|_E < \frac{\delta}{\sqrt{\nu}} \right)$$

Here $\mathbb{P}$ is the Wiener measure on $\Omega = C([0, 1], \mathbb{R}^d)$

$$\mathbb{P} = L(W).$$

Now, for any $\varphi \in \Omega$ we define

$$\mathbb{P}_\varphi = L(W + \varphi)$$

**Theorem** - Cameron–Martin

Assume that $\varphi \in W^{1, 2}([0, 1])$, with $\varphi(0) = 0$.

Then $\mathbb{P}$ and $\mathbb{P}_\varphi$ are equivalent.

and

$$\frac{d\mathbb{P}_\varphi}{d\mathbb{P}}(\omega) = \exp\left( -\frac{1}{2} \int_0^1 |\varphi'(s)|^2 \, ds + \int_0^1 \varphi(s) \, d\omega(s) \right)$$

Proof: We give an idea of the proof.
Let $0 \leq t_1 < t_2 < \ldots < t_n \leq T$
and $I_1, \ldots, I_n$ intervals.

Then,

$$P_{\Phi}(w(t_1) \in I_1, \ldots, w(t_n) \in I_n) = P(w(t_1) + \varphi(t_1) \in I_1, \ldots, w(t_n) + \varphi(t_n) \in I_n) =$$

$$= \left[2\pi \right]^{n t_1 (t_2 - t_1) \ldots (t_n - t_{n-1})} \frac{1}{(2\pi)^{t_n - t_{n-1}}} \exp \left( -\frac{1}{2} \sum_{k=1}^{n-1} \frac{(\xi_k - \xi_{k-1})^2}{t_k - t_{k-1}} \right) d\xi_1 \ldots d\xi_n$$

$$= \left[2\pi \right]^{n t_1 (t_2 - t_1) \ldots (t_n - t_{n-1})} \frac{1}{(2\pi)^{t_n - t_{n-1}}} \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \frac{(\varphi(t_k) - \varphi(t_{k-1}))^2}{t_k - t_{k-1}} \right) + \sum_{k=1}^{n} \frac{(\varphi(t_k) - \varphi(t_{k-1}))}{(w(t_k) - w(t_{k-1}))} \frac{1}{t_k - t_{k-1}} d\xi_1 \ldots d\xi_n$$

$$= \int_{\left\{ w(t_k) \in I_k, \ldots, w(t_n) \in I_n \right\}} dP(w)$$

Now, if $S = \{0 = \xi_1 < \ldots < \xi_N = T\}$
with $\xi_2 \in \{t_1, \ldots, t_n\}$
and $\varphi_\xi$ is the linear interpolation of
$\varphi(\xi_1), \varphi(\xi_2), \ldots, \varphi(\xi_N)$, we get

$$P_{\varphi_\xi}(w(t_1) \in I_1, \ldots, w(t_n) \in I_n) =$$
The general formula follows by taking the limit, as $16\varepsilon \rightarrow 0$.

Now, let us go back to the proof of Step I.

\[ P(\| \sqrt{\varepsilon} W - \varphi \|_E < \delta) = P(\| W - \frac{\varphi}{\sqrt{\varepsilon}} \|_E < \frac{\delta}{\sqrt{\varepsilon}}) \]

\[ = \int P_{-\varphi/\sqrt{\varepsilon}}(B(0, \frac{\delta}{\sqrt{\varepsilon}})) \, \frac{d\varepsilon}{\sqrt{\varepsilon}} \]

\[ = \int \exp\left(-\frac{1}{2} \int_0^T \| \varphi'(s) \|^2 \, ds - \frac{1}{2} \sqrt{\varepsilon} \right) < \varphi'(s), \, dw(s) > \, \frac{d\varepsilon}{\sqrt{\varepsilon}} \]

\[ = \exp\left(-\frac{1}{2} \int_0^T \| \varphi'(s) \|^2 \, ds \right) \int \exp\left(-\frac{1}{2} \sqrt{\varepsilon} \right) < \varphi'(s), \, dw(s) > \, \frac{d\varepsilon}{\sqrt{\varepsilon}} \]

Due to the
Symmetry of $P$

\[
\geq \exp\left( -\frac{1}{2\varepsilon} \int_0^\infty P(\|W\| < \frac{\varepsilon}{\sqrt{2}}) \right)
\]

Now

\[
P(\|W\| < \frac{\varepsilon}{\sqrt{2}}) = 1 - P(\|W\| \geq \frac{\varepsilon}{\sqrt{2}})
\]

\[
\geq 1 - \frac{\varepsilon}{\sqrt{2}} \mathbb{E}\left( \sup_{t \in [0, \pi]} |W(t)|^2 \right) \geq 1 - \frac{2C_1}{\varepsilon^2}
\]

This implies

\[
P(\|W - \phi\| < \delta) \geq \exp\left( -\frac{1}{3} \mathbb{E}(\tilde{\phi}) \right) \left( 1 - \frac{C_1}{\delta^2} \right)
\]

Now, given $\gamma > 0$

\[
\exists \epsilon_0 \text{ such that } \\
\left( 1 - \frac{C_1}{\delta^2} \right) \geq e^{-\gamma \epsilon_0}
\]

and then our theorem follows.

\[
\text{STEP II} \\
\text{Given } \phi \in \tilde{E}, \delta > 0, \gamma > 0, \epsilon > 0 \exists \epsilon_0 > 0 \text{ such that } \\
\text{Mes}(B^c(I^r, \delta)) \leq \exp\left( -\frac{1}{3}(r - \delta) \right)
\]

Proof. We denote by $W^H(t)$ the linear interpolation of $W(t)$ in $[0,1]$ with steps
We want to estimate

$$P(\text{dist}_E(\mathbb{V}^E, I') \geq \delta) -$$

we have

$$P(\text{dist}_E(\mathbb{V}^E, I') \geq \delta) =$$

$$P(\text{dist}_E(\mathbb{V}^E, I') \geq \delta, \quad \forall e \in I^E) +$$

$$P(\text{dist}_E(\mathbb{V}^E, I') \geq \delta, \quad \exists e \in I^E)$$

$$\leq P(\|W - W_m\|_E \geq \delta \sqrt{\varepsilon}) + P(\exists e \in I^E) = J_1 + J_2.$$

Now

$$J_2 = P(\forall e \in I^E) = P(\forall e \in I^E) > r$$

$$= P(\forall e \in I^E) > r$$

$$= P(\forall e \in I^E) > r$$

$$= P(\sum_{j=0}^{m-1} \|W(t_{j+1}) - W(t_j)\|^2 \geq r)$$

$$= P(\sum_{j=0}^{m-1} Y_j^2 \geq r / \varepsilon)$$

where $Y_0, \ldots, Y_{m-1}$ are i.i.d. with $Y \sim \mathcal{N}(0,1)$

This implies $m \leq (0, 1/2)$

$$J_2 = P(\exp \sum_{j=0}^{m-1} Y_j^2 \geq \exp(\frac{r}{\varepsilon}))$$

$$\leq e^{-r / \varepsilon} \left[\mathbb{E}(\exp(\lambda Y^2))\right]^m = e^{-\frac{2r}{\varepsilon}} (1 - 2\alpha)^{-m / 2}$$

and we choose $\overline{\alpha}$ and $\overline{m}$.
Such that
\[ e^{-\frac{r}{3(1-2\epsilon)}} (1-2\epsilon)^{-m/12} \leq \frac{1}{2} e^{-\frac{(1-\epsilon)}{3}} \quad \text{for } m \geq M. \]

For \( J_4 \) we have
\[ J_4 = \mathbb{P}(\|W-W_m\|_E \geq \frac{\delta}{\sqrt{3}}) \leq m \mathbb{P}\left( \sup_{t \in [0,1]} |W(t) - \text{Smw}(\frac{t}{m})| \geq \frac{\delta}{\sqrt{3}} \right). \]

Due to this, we have that the family \( \{\mu_\epsilon \} = \{ \mathbb{P}(W) \} \) satisfies a LDP in \( E = C_0([0,1]; \mathbb{R}^d) \) with action functional
\[ I(\phi) = \frac{1}{2} \int_0^1 |\phi'(s)|^2 ds. \]

**EXAMPLE.**

Let \( K = \{ \phi \in C([0,1]) : \frac{1}{2} \int_0^1 \phi^2(s) ds \geq 1 \} \)

Then \( \inf_{\phi \in K} I(\phi) = \frac{\pi^2}{4} \).

Actually, if we define
\[ A\phi = \phi'' \quad \forall \phi \in D(A) = \{ \phi \in W^{1,2}(0,1) : \phi'(0) = 0, \phi'(1) = 0 \} \]
Then $A$ is self adjoint in $L^2(0,1)$ and $\phi_k(x) = \sqrt{2} \sin \left( \pi k x \right)$ is a c.o.s. in $L^2(0,1)$ which diagonalizes $A$.

Then
\[
\min_K \int_0^1 |\phi'(s)|^2 \, ds = \min_{\|\phi\|_2 \geq 1} \langle A\phi, \phi \rangle = \frac{\pi^2}{4}
\]

Therefore, since
\[
\inf_K \int_0^1 |\phi'(s)|^2 \, ds = \inf_K \int_0^1 |\phi(s)|^2 \, ds
\]

we have
\[
\mathbb{P} \left( \int_0^1 |w(s)|^2 \, ds \geq \frac{1}{2} \right) \leq e^{-\frac{1}{2} \frac{\pi^2}{8}}
\]

Now, we consider the equation
\[
\begin{cases}
\frac{dX(t)}{dt} = b(X(t))dt + \sqrt{d} \, dW(t) \\
X(0) = x
\end{cases}
\]

For $T > 0$, we take
\[
\mathbb{E} := C_{T,x} = \{ \phi : [0,T] \to \mathbb{R}^d \text{ continuous} \}
\]

For $T > 0$, we take
\[
\mathbb{P}_x = \mathcal{L}(X^x) \text{ in } C_{T,x}
\]
We are interested in the validity of the LDP for this family.

Now, we consider the following mapping
\[ \phi(x) : C_{t,1} \to C_{1,t} \]
\[ \phi(t) \mapsto \phi_x(t) = \text{solution of the integral equation} \]
\[ \psi(t) = x + \int_0^t b(\psi(s)) \, ds + \psi(0). \]

As \( t \in C_{1,t} \), there exists a solution \( \psi \) such that \( \phi_x \) is well defined.

Moreover, \( \phi_x(\phi_1) - \phi_x(\phi_2) \) solves
\[ \psi(t) = \int_0^t \left[ b(\phi(s)) - b(\phi_2(s)) \right] \, ds + (\phi_1 - \phi_2) \psi(0) \]
so that
\[ |\psi(t)| \leq \int_0^t M |\psi(s)| \, ds + |\phi_1 - \phi_2|. \]

And hence, by the Gronwall lemma,
\[ |\psi| \leq e^{mt} |\phi_1 - \phi_2|. \]

This means that \( \phi_x \) is Lipschitz-continuous.
Next, it is immediate to check that
\[ \phi_{-1}(y) = y(4) - x - \int_{0}^{x} \lambda(y(s)) \, ds. \]

The action functional for \( \{ \mu_{x} \} = \{ \nu \} \) is
\[ I(\nu) = \frac{1}{2} \int_{0}^{T} |\nu'(13)|^2 \, ds. \]

Now we define
\[ S_{t,x}(\psi) = I_{T}(\phi_{t}(x)) \quad \forall x \in C_{0,t}, \]
\[ = \frac{1}{2} \int_{0}^{T} |y'(t) - b(y(t))|^2 \, dt. \]

**THEOREM** \( \{ \nu_{x} \} \) satisfies a LDP in \( C_{0,t} \) with action functional \( S_{t,x} \).

**Proof.** This follows from the so-called CONTRACTION PRINCIPLE.

Let \( \phi : E \rightarrow F \) be an homeomorphism \( E, F \) Banach spaces.

\( \{ \mu_{x} \} \) probabilities in \( E \)
\[ \nu_{x} = \mu_{x} \circ \phi^{-1} \]
\[ J(y) = I(\phi^{-1}(y)) \quad \forall y \in F. \]
$J : F \to [0, \infty]$ is lower semi-continuous

$\neq \infty$ and

$\{ J \leq r \} = \{ y \in F : J(y) \leq r \}$ is compact

Theorem: Assume that $\{ F \}$ satisfies a LDP in $E$

with action functional $I$.

Then $\{ F \}$ satisfies a LDP in $F$

with action functional $J$.

Proof: Let $A \subset F$ open $\implies \phi^{-1}(A) \subset F$ open

$\liminf_{S \to 0} S \log \mathbb{P}_F(A) = \liminf_{S \to 0} \mathbb{E} \log \mathbb{P}_F(\phi^S(A))$

$\geq - \inf_{x \in \phi^{-1}(A)} I(x) = - \inf_{y \in A} J(y)$

Analogously for $C$ closed in $F$.

Remark:

(Trivial but important)

$s_{\cdot, x}(\psi) = 0 \iff \psi(t) = X_0^x(t)$

Actually $s_{\cdot, x}(\phi) = \frac{1}{2} \int_0^T (\dot{\phi}(s) - \phi(\theta(s)))^2 \, ds \leq 0$
\( \varphi'(s) = h(\varphi(s)) \)

(\( \varphi(0) = x \))

(\( \varphi(t) = X^x(t) \))

(by uniqueness)