HOMEWORK ASSIGNMENT 3

DUE SEPT. 30

(1) A function is called Lipschitz-continuous with constant $M$ on $[a, b]$ if for any $x, y \in [a, b]$

$$|f(x) - f(y)| \leq M|x - y|.$$ 

Note that a Lipschitz-continuous function is not necessarily differentiable on $[a, b]$. 

Consider the linear interpolation for Lipschitz-continuous functions with constant $M$ on the interval $[a, a + h]$ at the nodes $a$ and $a + h$.

(a) Suppose we are given: $f(a) = f_0$, $f(a + h) = f_1$. Write an expression for the linear interpolant of $f(x)$ on the interval $[a, a + h]$.

(b) Give an example of a Lipschitz-continuous function $f(x)$ with constant $M$ for which the error of this linear interpolation will be maximal possible.

(c) Find an error bound of the linear interpolation for Lipschitz-continuous functions with constant $M$ on the interval $[a, a + h]$.

(2) Prove the following

**Theorem 1.** Let the real function $f$ be $n + 1$ times differentiable on the interval $[a, b]$, and consider $m + 1$ support points $x_i \in [a, b], \quad x_0 < x_1 < \ldots < x_m.$

If the polynomial $P(x)$ is of degree at most $n$,

$$n + 1 = \sum_{i=0}^{m} n_i,$$

and satisfies the interpolation conditions

$$P^{(k)}(x_i) = f^{(k)}(x_i), \quad i = 0, 1, \ldots, m, \quad k = 0, 1, \ldots, n_i - 1,$$

then for every $y \in [a, b]$ there exists $\xi \in I[x_0, \ldots, x_m, y]$ such that

$$f(y) - P(y) = \frac{\omega(y) f^{(n+1)}(\xi)}{(n+1)!},$$

where

$$\omega(x) \equiv (x - x_0)^{n_0}(x - x_1)^{n_1}\ldots(x - x_m)^{n_m}.$$ 

(3) Let $p_3(x)$ be the second-degree polynomial which interpolates the given values $y_j = f(x_j)$ at the three distinct nodes, $x_1 < x_2 < x_3$.

(a) We are now given the additional value of the derivative, $y'_3 = f'(x_3)$. Let $p_4(x)$ be the polynomial which interpolates the three values, $p_4(x_j) = y_j$, $j = 1, 2, 3$, and the additional derivative $p'_4(x_3) = y'_3$. Then $p_4(x)$ has the form $p_4(x) = p_3(x) + q(x)$.

Write an explicit expression for $q(x)$.

(b) Give a formula for the interpolation error $f(x) - p_4(x)$ using the derivatives of $f$.

(c) Suppose we change the value at node $x_1$ from $y_1$ to $\tilde{y}_1 = y_1 + \epsilon$, while the data at the other nodes, $y_2, y_3$ and $y'_3$ remains unchanged. Express the corresponding interpolating polynomial $\tilde{p}_4(x)$ in the form $\tilde{p}_4(x) = p_4(x) + er(x)$ for an appropriate $r(x)$. 