We want to approximate the integral $\int_a^b f(x)w(x)dx$ where $w(x)$ is a weight function.

(a) Define an $n$-point using a quadrature rule of the form $Q(f) = \sum_{j=1}^n f(x_j)w_j$ based on interpolating $f$ with a polynomial. Write down the formulas for the weights $w_j$, $j = 1, \ldots, n$.

(b) What is its minimal exactness?

(c) Suppose $w(x) = 1$ and $[a, b] = [0, 1]$. Using Matlab, Maple, or Mathematica, show that if $Q(f)$ is a Newton-Cotes quadrature, the weights are not necessarily all positive for large $n$’s. (Find an example of $n$ where some of the weights are negative.)

Suppose you want to compute the integral

$$
\int_{-1}^1 e^{-\frac{1}{1-x^2}}dx
$$

using the composite trapezoidal rule. How fast the integration error should decay with the number of subintervals? Use the Euler-Maclaurin summation formula to justify your expectation.

Let $w(x)$ be a weight function and $\{p_n\}$ be a family of monic polynomials orthogonal w.r.t. the corresponding inner product. As we have shown in the class, $p_n$’s satisfy the Three-Term Recurrence Relationship (TTRR) of the form

$$
x p_0 = p_1 + B_0 p_0, \\
x p_k = p_{k+1} + B_k p_k + A_k p_{k-1}, \quad k = 1, 2, \ldots, \quad \text{where}$$

$$
A_k = \frac{\|p_k\|^2}{\|p_{k-1}\|^2}, \quad k \geq 1, \quad B_k = \frac{\langle p_k, p_k \rangle}{\|p_k\|^2}.
$$

However, to build the Gaussian quadrature, we will need the TTRR coefficients for the corresponding orthonormal set of polynomials $\{p_n\}$. Show that they also satisfy a TTRR of the form

$$
x p_0 = \alpha_1 p_1 + \beta_0 p_0, \\
x p_k = \alpha_{k+1} p_{k+1} + \beta_k p_k + \alpha_k p_{k-1}, \quad k = 1, 2, \ldots.
$$

Express $\alpha_k$’s and $\beta_k$’s via $A_k$’s and $B_k$’s.

(b) Often for a given “classic” weight function the “classic” orthogonal polynomials are neither monic nor orthonormal (e.g. Chebyshev, Legendre, etc.), and satisfy a TTRR of the form

$$
x P_0 = a_0 P_1 + b_0 P_0, \\
x P_k = a_k P_{k+1} + b_k P_k + c_k P_{k-1}, \quad k = 1, 2, \ldots.
$$

Find the TTRR for the corresponding set of orthonormal polynomials of the form of Eq. (1). Express $\alpha_k$’s and $\beta_k$’s via $a_k$’s, $b_k$’s, and $c_k$’s.

(4) We want to approximate the integral $I(f) = \int_0^1 f(x)\sqrt{x}dx$ by a quadrature formula $Q_n(f) := \sum_{j=1}^n w_j f(x_j)$ at the nodes $\{x_j\}_{j=1}^n$ using the weights $\{w_j\}$. Assume that there exists a polynomial $p$ for which the quadrature $Q_n$ is exact: $Q_n(p) = I(p)$. Prove that if $w_j \geq 0$ then

$$
|I(f) - Q_n(f)| \leq \frac{4}{3} \max_{x \in [0,1]} |f(x) - p(x)|.
$$

**Hint:** verify that $\sum_{j=1}^n w_j = \int_0^1 \sqrt{x}dx = \frac{2}{3}$ and that $I(f) - Q_n(f) = I(f-p) + Q_n(p-f)$.

(5) Suppose $f(x)$ is a Lipschitz-continuous function with constant $M$ on $[0,2h]$. Note that $f(x)$ is not necessarily differentiable. Suppose you are given $f(0)$, $f(h)$, and $f(2h)$. Find the exact error bounds for its quadrature over $[0,2h]$ for the composite trapezoidal rule with subinterval $h$, and the Simpson rule. Which of these two rules is it safer to use?