Fall 2008 - Math 462 Section 0101
Partial Differential Equations for Scientists and Engineers
Practice Problems

1. Solve the IBVP

$$\begin{cases} u_{tt} - 9u_{xx} = 0\,, \qquad 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0\,, \qquad t > 0 \\ u(x,0) = 5\sin(3x)\,, \qquad 0 < x < \pi \\ u_t(x,0) = 1\,, \qquad 0 < x < \pi \end{cases}$$

2. Consider the following damped wave equation on $(0,1)$:

$$u_{tt} - 4u_{xx} + au_t = 0$$

with boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 0.$$

(a) Show the identity

$$\int_0^1 \frac{1}{2}|u_t(x,t)|^2 + 2|u_x(x,t)|^2\,dx + a\int_0^t\int_0^1 |u_t(x,s)|^2\,dx\,ds = \int_0^1 \frac{1}{2}|u_t(x,0)|^2 + 2|u_x(x,0)|^2\,dx$$

(b) Solve the PDE by separation of variables when $0 < a < 2\pi$ and the initial conditions are given by

$$u(x,0) = 0 \qquad u_t(x,0) = \phi(x).$$

3. Solve

$$\sqrt{1+x^2}\,u_x + xy\,u_y = -\frac{1}{\sqrt{1+x^2}}u$$

$$u(0,y) = y/e$$

4. Consider the PDE

$$u_{tt} + 2au_t + a^2u = c^2u_{xx}$$

where a and c are positive constants.

(a) Let $v(x,t) = e^{at}u(x,t)$; show that v satisfies the wave equation $v_{tt} = c^2v_{xx}$.

(b) Using d'Alembert formula, find the solution $u(x,t)$ of the PDE above for $t>0$, $-\infty<x<\infty$ with initial conditions

$$u(x,0)=f(x), \quad u_t(x,0)=0.$$

5. Solve Laplace's equation $\Delta u=0$ in the rectangle $0<x<1$, $0<y<1$ with boundary conditions

$$u_x(0,y)=0, \quad u_x(1,y)=0$$
$$u(x,0)+u_y(x,0)=0, \quad u_y(x,1)=1$$

6. Find the solution of $\Delta u=0$ in the annulus $\{(r,\theta)\,;\,1<r<2\}$ with mixed boundary conditions

$$u_r(1,\theta)=0 \qquad u(2,\theta)=2$$

7. Consider the solution of

$$u_t-u_{xx}+au=0 \qquad 0<x<2 \quad t>0$$

with the boundary condition

$$u(0,t)=0, \qquad u(2,t)=\sin t$$

and initial condition

$$u(x,0)=1.$$

(a) Find the equation and boundary conditions satisfied by $v(x,t)=e^{at}u(x,t)$

(b) Show that $|u(x,t)|\le 1$ for all $0<x<2$, $t>0$.
Hint: First show that $|v(x,T)|\le e^{aT}$ for all $0<x<2$, $T>0$ (apply the maximum principle for v in $Q_T=\{0<x<2\,,\ 0<t<T\}$).