Fall 2008 - Math 462 Section 0101
Partial Differential Equations for Scientists and Engineers
Homework #5 - Due Thursday Oct 9th

1. (25pt) The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

   (a) Verify that $u(x,t) = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2\,,\ 0 \le t \le 1\}$. Why does this contradict the maximum principle?

   (b) Where precisely does the proof of the maximum principle break down for this equation?

2. (25pt) Compute $\int_0^\infty e^{-x^2}\,dx$. (**Hint:** Notice that

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy = \int_0^\infty \int_0^\infty e^{-(x+y)^2}\,dx\,dy,$$

   and use polar coordinate to compute this last integral).

3. (25pt) Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty$$

   with $u(x,0) = \phi(x)$, where $b > 0$ is a constant. (**Hint:** Make the change of variable $u(x,t) = e^{-bt}v(x,t)$).

4. (25pt) Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for } -\infty < x < \infty$$

   with $u(x,0) = \phi(x)$, where V is a constant. (**Hint:** Substitute $y = x - Vt$.)