Math 340 M. Brin Fall 2009 EXAM 1

Each problem is 20 points. Solve each problem on a separate answer sheet. Show all work. Justify and mark clearly all answers, unless instructed otherwise.

1. The triangle axiom (for vectors) says that $\|x+y\| \leq \|x| + \|y\|$ for every two vectors x and y.
Prove that $\|u-v-w\| \leq \|u\| + \|v\| + \|w\|$ for every three vectors u,v,w.

Solution.
By the triangle inequality and since $\|-x\| = \|x\|$, we have
$\|u-v-w\| = \|u+(-v)+(-w)\| \leq \|u\|+\|-v\|+\|-w\| = \|u\|+\|v\|+\|w\|$.

2. Let P be the plane passing through the points $(2,0,0)$,$(0,2,0)$,$(0,0,4)$.
a) Find an equation for P in the form $ax+by+cz=d$.
b) Find the distance from the origin $(0,0,0)$ to P.

Solution.
Looking for an equation in the form $ax+by+cz=1$ we obtain $a=1/2$,$b=1/2$,$c=1/4$; $\frac{1}{2}x+\frac{1}{2}y+\frac{1}{4}z=1$, or $2x+2y+z=4$. After normalization, $\frac{2}{3}x+\frac{2}{3}y+\frac{1}{3}z=\frac{4}{3}$. The distance is $|4/3| = 4/3$.

3. True or false questions. "True" means it is always true. "False" means there is at least one case (example) when it is false. Each correct answer is $+4$ points, each wrong answer is -4 points, no answer – 0 points. No explanation or justification is required.

a) Let $m<n$. For every matrix A with m rows and n columns, the homogeneous system of equations $Ax=0$ has a nontrivial (nonzero) solution.
b) Let $m<n$. For every matrix A with m rows and n columns and any vector $b \in \mathbb{R}^n$, the nonhomogeneous system of equations $Ax=b$ has a solution.
c) For every two $n\times n$ matrices A and B, if $AB=O$, then $BA=O$.
d) For every two $n\times n$ matrices A and B, if $AB=I$, then $BA=I$.
e) The determinant of every *antisymmetric* 13×13 matrix is 0.

Solution.
a) Since there are fewer equations than unknowns and the right hand side is 0, there are infinitely many solutions.
b) Since the right hand side is not 0, there may be no solutions.
c) It is possible even for 2×2 matrices that $AB = O$ and $BA \neq O$.
d) Since $AB = I$ we have that $B = A^{-1}$ (or $A = b^{-1}$). Therefore $BA = I$.
e) We have $\det A = \det A^t = (-1)^{13} \det A = -\det A$. Therefore $\det A = 0$.

4. Suppose that A is an $n \times n$ matrix such that $A^{340} = O$.
Prove that $\det A = 0$.

Solution.
Since $A^{340} = O$ we have that $\det A^{340} = (\det A)^{340} = 0$. Therefore $\det A = 0$.

5. Let $\mathbf{n}$ be the unit vector $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Find the matrix corresponding to the orthogonal projection of $\mathbb{R}^3$ onto the line $\{t\mathbf{n}\}$ spanned by $\mathbf{n}$.

Solution.
Let the matrix be A. Then Ae_1 (the first column of A) is the projection of e_1 onto $\{t\mathbf{n}\}$, which is $(\mathbf{n} \cdot e_1)\mathbf{n} = (1/3, 1/3, 1/3)$. Same for the other columns.