Math 340 M. Brin Fall 2009 EXAM 2

Each problem is 20 points. Solve each problem on a separate answer sheet. Show all work. Justify and mark clearly all answers.

1. Let $f\colon \mathbb{R} \to \mathbb{R}^n$, be a contiuosly differentiable curve in $\mathbb{R}^n$. Suppose $\|f(t)\| = 340$ for all t. Prove that $f(t) \cdot f'(t) = 0$ for all t (i.e., $f(t) \perp f'(t)$).
Solution. Since $\|f(t)\|^2 = \text{const}$, its derivative is 0. We have $0 = \frac{d}{dt}\|f(t)\|^2 = 2f(t) \cdot f'(t)$.

2. Consider the one sheet hyperboloid $x^2 + y^2 - z^2 = 1$ in $\mathbb{R}^3$. Find an equation of the tangent plane to this surface at the point $x = y = z = 1$ in the form $ax + by + cz = d$.
Solution. $f(x, y, z) = x^2 + y^2 + z^2$, $\nabla f = (2x, 2y, -2z)$, $\nabla f(1, 1, 1) = (2, 2, -2)$. Hence the tangent plane is given by $2x + 2y - 2z = 2$ or $x + y - z = 1$.

3. The equations $\sin(x + u) - \cos(y + v) + xy - u + uv = 1$ and $u + v + x + y - ux - vy = 5$ implicitly define x and y as functions of u and v. The point $u = 2$, $v = 1$, $x = -2$, $y = -1$ satisfies the equations.
Find $\dfrac{\partial x}{\partial u}(2, 1)$ (i.e., for $u = 2$, $v = 1$, $x = -2$, $y = -1$).
Solution. Differentiate both equations with respect to u to obtain a system of two equations for $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$. As it happens only the second equation is needed: $1 + \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} - x - u\frac{\partial x}{\partial u} - v\frac{\partial y}{\partial u} = 0$; for the given values of the variables we get $1 + \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} - 2\frac{\partial x}{\partial u} - (-2) - 1 \cdot \frac{\partial y}{\partial u} = 0$ and $\frac{\partial x}{\partial u} = 3$.

4. Find the minimum and maximum values of $f(x, y) = x^2 + 4xy + y^2$ in the region $x^2 + y^2 \le 1$.
Solution. The only critical point is the origin $(0, 0)$ and it is a saddle. Using Lagrange multipliers we get: $x^2 + y^2 = 1$, $2x + 4y = \lambda 2x$, $2x + y = \lambda 2y$; after simplifying: $y = \frac{\lambda - 1}{2}x$, $x = \frac{\lambda - 1}{2}y$. There are 4 solutions: $x = y = \pm\frac{1}{\sqrt{2}}$ (this gives a maximum value of 3) and $x = -y = \pm\frac{1}{\sqrt{2}}$ (this gives a minimum value of -1).

5. Let R be the region in the xy-plane bounded by the x-axis and the arc of the sine curve $y = \sin x$, $0 \le x \le \pi$. Let D be the region in 3-space which is bounded from below by R (in the xy-plane) and from above by the graph of $z = x + 2y$. Find the volume of D.

Solution. $\displaystyle\int_0^\pi dx \int_0^{\sin x} dy (x + 2y) = \int_0^\pi x \sin x + \sin^2 x dx = \frac{3}{2}\pi$