(1) Find \( \lim_{n \to \infty} n^{205} \left( \sqrt{n^{410} + 2} - \sqrt{n^{410} - 1} \right) \).

Multiply the numerator and denominator by \((\sqrt{n^{410} + 2} + \sqrt{n^{410} - 1})\)
\[
n^{205} \left( \sqrt{n^{410} + 2} - \sqrt{n^{410} - 1} \right) = n^{205} \frac{3n^{205}}{n^{410} + 2 + n^{410} - 1} = \frac{3n^{205}}{2n^{410} + 2n^{410} - 1} \to \frac{3}{2}.
\]

(2) Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a strictly increasing function defined for all real numbers.

Prove that the limit from the right \( \lim_{x \to 0^+} f(x) \) exists.

Let \( S = \{ f(x) : x > 0 \} \). The set \( S \) is not empty (contains \( f(1) \)) and bounded from below (for example, by \( f(0) \)). Therefore, by the completeness axiom, \( \alpha = \inf S \) exists. Claim: \( \lim_{x \to 0^+} f(x) = \alpha \). To prove this fix any \( \epsilon > 0 \). Since \( \alpha + \epsilon \) is not a lower bound for \( S \), there is \( \delta > 0 \) such that \( f(\delta) < \alpha + \epsilon \). Since \( f \) is increasing, \( f(x) < f(\delta) < \alpha + \epsilon \) for each \( x \) with \( 0 < x < \delta \). Hence, \( 0 \leq f(x) - \alpha < \epsilon \) for \( 0 < x < \delta \).

(3) A subset \( S \subseteq \mathbb{R} \) is open if for every \( x \in S \) there is \( \epsilon > 0 \) such that \((x - \epsilon, x + \epsilon) \subseteq S \).

Let \( S_1, S_2, \ldots, S_k \) be open subsets of \( \mathbb{R} \). Prove that the intersection \( \bigcap_{i=1}^k S_i \) is open.

Let \( x \in \bigcap_{i=1}^k S_i \). Then \( x \in S_i \) for each \( i \) and there is \( \epsilon_i > 0 \) such that \((x - \epsilon_i, x + \epsilon_i) \subseteq S_i \). Choose \( \epsilon = \min_{i=1,2,\ldots,k} \epsilon_i \). Then the interval \((x - \epsilon, x + \epsilon)\) is contained in each \( S_i \), and hence in the intersection.

(4) Suppose that \( -x^4 \leq f(x) \leq x^4 \) for all \( x \in \mathbb{R} \).

Prove that \( f \) is differentiable at 0 and that \( f'(0) = 0 \).

Observe that \( f(0) = 0 \). We have: \( \left| \frac{f(x) - f(0)}{x - 0} \right| \leq \frac{|f(x)|}{|x|} \leq |x|^3 \to 0 \) as \( x \to 0 \).

Hence \( f'(0) = 0 \).

(5) Suppose that a function \( g : \mathbb{R} \to \mathbb{R} \) is differentiable at \( x = 0 \). Also, suppose that \( g(1/n) = 0 \) for each natural number \( n \).

Prove that a) \( g(0) = 0 \), b) \( g'(0) = 0 \).

a) Since \( g \) is differentiable at 0, it is continuous at 0. Therefore, since \( \lim_{n \to \infty} 1/n = 0 \), we have \( f(0) = \lim_{n \to \infty} g(1/n) = \lim_{n \to \infty} 0 = 0 \).

b) Since \( g'(0) \) exists, \( \lim_{n \to \infty} \frac{g(x_n) - g(0)}{x_n - 0} = g'(0) \) for every sequence \( \{x_n\} \) with \( \lim_{n \to \infty} x_n = 0 \). Observe that \( \frac{g(1/n) - g(0)}{1/n - 0} = g'(0) \) for each \( n \). Hence, \( \lim_{n \to \infty} \frac{g(1/n) - g(0)}{1/n - 0} = 0 \) and \( g'(0) = 0 \).