1. A 100-gallon tank initially contains 50 gallons of water with a total of 10 pounds of salt dissolved in it. A drain is opened in the bottom which lets out 1 gallon of solution per minute. Simultaneously, salt solution begins to be added at 2 gallons per minute with a concentration of 2 pounds per gallon.

Find the concentration of salt in the tank at the moment it becomes full.

**Solution.** Let $V(t)$ be the volume of solution in the tank and $s(t)$ the amount of salt in the tank. Then $V(t) = 50 + t$, $s(0) = 10$ and $s'(t) = 4 - \frac{s}{50+t}$. $(s(50 + t))' = 200 + 4t; s(50 + t) = 200t + 2t^2 + C; C = 500; s(50) = 155; s(50)/100 = 1.55$.

2. For the differential equation $y'' - 2y' + 10y = 0$
   a) find the general solution (in real form),
   b) find the solution such that $y(0) = 2$, $y'(0) = 5$ (in real form).

**Solution.** a) $r = 1 \pm 3i, y = e^x(C_1 \cos 3x + C_2 \sin 3x)$.
   b) Since $y(0) = 2$ we have $C_1 = 2$. Substitute $y = e^x(2 \cos 3x + C_2 \sin 3x)$ in the equation to get $C_2 = 1$. $y = e^x(2 \cos 3x + \sin 3x)$.

3. a) Find the general solution of $y'' - 3y' + 2y = 0$.
   b) Find an appropriate form for a solution of $y'' - 3y' + 2y = 3x^2 \sin x - 7x \cos x + 4xe^{-x} - x^2 e^{-2x}$

**Solution.** a) $r^2 - 3r + 2 = 0, r = 1, 2, y = C_1 e^x + C_2 e^{2x}$.
   b) $y = (A_1 x^2 + A_2 x + A_3) \cos x + (B_1 x^2 + B_2 x + B_3) \sin x + (C_1 x + C_2) e^{-x} + (D_1 x^2 + D_2 x + D_3) e^{-2x}$.

4. a) Find the general solution of $x'' + 2x' + 2x = 0$ (in real form).
   b) Find the steady state of $x'' + 2x' + 2x = \cos t$ in the form $A \cos(\omega t - \alpha)$.

**Solution.** a) $r^2 + 2r + 2 = 0; r = -1 \pm i; x(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$
   b) Substitute $y = a \cos t + b \sin t$ into the equation to get $\frac{1}{5}(\cos t + 2 \sin t)$; $y = \frac{1}{\sqrt{5}} \cos(t - \alpha)$, where $\cos \alpha = \frac{1}{\sqrt{5}}, \sin \alpha = \frac{2}{\sqrt{5}}$.

5. a) Find the general solution of the system
   \[ \begin{align*}
   \dot{x} &= 8x - 3y, \\
   \dot{y} &= 10x - 3y.
   \end{align*} \]
   b) Find the particular solution satisfying $x(0) = 0$, $y(0) = 1$.

**Solution.** a) $\lambda^2 - 5\lambda + 6 = 0$, $\lambda = 2, 3$. For $\lambda = 2$, an eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
   For $\lambda = 3$, an eigenvector is $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$. \( \begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \).
   b) The initial conditions give $C_1 = 3, C_2 = -1$; \( \begin{pmatrix} x \\ y \end{pmatrix} = 3e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - e^{3t} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \).