Shifts of finite type and sofic shifts: from $\mathbb{Z}$ to $\mathbb{Z}^d$

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Outline of the talk

I. Background on shifts

II. Overview: Z^d SFTs and sofic shifts, $d=1$ vs. $d\geq 2$

III. Specific contrasts, $d=1$ vs. $d\geq 2$ (and an excursion to cellular automoata

IV. A hint of proof?

I. Background on shifts

Z^d Subshifts

Notations: given positive integers d, N:

- $\mathcal{A} := \{0, 1, \ldots, N-1\}$, a finite alphabet
- An element of $\mathcal{A}^{\mathbb{Z}^d}$ is pictured as a way of filling the Z^d lattice with symbols
- For $d = 1$, x in $\mathcal{A}^{\mathbb{Z}^d}$ is $x = \ldots x_{-1}x_0x_1 \ldots$, with all x_i in $\mathcal{A}$ and $x_i = x(i)$
- For $d = 2$, x in $\mathcal{A}^{\mathbb{Z}^d}$ is a planar array of symbols, etc.

- $\mathcal{A}^{\mathbb{Z}^d}$ is a metric space, $\text{dist}(x,y) = \frac{1}{k+1}$, where $k = \min\{||v|| : x(v) \neq y(v)\}$

- There is a shift action σ of $\mathbb{Z}^d$ on $\mathcal{A}^{\mathbb{Z}^d}$ by homeomorphisms. For $v \in Z^d$, the shift homeomorphism σ^v is defined by $(\sigma^v x)(u) = x(u+v)$, $u \in \mathbb{Z}^d$

- For example, for $d = 1$ and $y = \sigma^1 x$: if $\ldots x_{-1}x_0x_1 \cdots = \ldots 003\ldots$, then $\ldots y_{-2}y_{-1}y_0 \cdots = \ldots 003\ldots$.

- If Y is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, then $(Y, \sigma|_Y)$ is a *subshift* (or just *shift*).

- For any subshift Y, there exists a set $\mathcal{L}$ of finite configurations such that Y is the set of all x in $\mathcal{A}^{\mathbb{Z}^d}$ satisfying:
  for every finite subset C of Z^d and $u \in Z^d$,

  $$x|_{(u+C)} \notin \mathcal{L}.$$

  If it is possible to choose $\mathcal{L}$ to be some finite set $\mathcal{F}$ then Y is a *shift of finite type* (SFT).

- Example: $d = 1$, $\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{00\}$.

Block codes

Let Y be a subshift on alphabet $\mathcal{A}$.
Let Y' be a subshift on alphabet $\mathcal{A}'$.
Let $U_n = \{v \in Z^d : ||v|| \leq n\}$.

- A block code is a function $\phi : Y \to Y'$ for which there exists $\Phi : \mathcal{A}^{U_n} \to \mathcal{A}'$ with $(\phi x)(v) = \Phi(x|_{v+U_n})$ for all $v \in \mathbb{Z}^d$.

- Example: $Y = Y' = \{0, 1\}^Z$ and $(\phi x)_i = x_i + x_{i+1}$ (mod 2)

- ϕ is a 1-block code if $(\phi x)(v) = \Phi(x|_v)$

- The continuous shift-commuting maps between subshifts are exactly the block codes.

- A surjective block code is a quotient map or factor map of subshifts. A bijective block code is a topological conjugacy, or isomorphism, of subshifts.

- For $d = 1$ and $\mathcal{A} = \{0, \ldots, N-1\}$ and B an $N \times N$ zero-one matrix, define
  $X_B = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall i, B(x_i, x_{i+1}) = 1\}$.
  Here $\sigma|_B = \sigma_B$ is a Z-SFT (a "vertex shift").

- Any Z-SFT is topologically isomorphic to X_B for some matrix B.

- Likewise for $d > 1$, a Z^d SFT is topologically conjugate to one defined by d $N \times N$ matrices, which give the allowed transitions in each of the d dirctions.

Sofic Shifts

- A sofic shift is a subshift which is a quotient of an SFT.

- For example: SFT $= X_{\mathcal{F}}$, on alphabet $\{a, b, c\}$ with $\mathcal{F} = \{ba, bb, cc\}$. (I.e.a point in X is an arbitrary concatatenation of the words a and bc.) Define Y as the image under the one-block code $a \mapsto 1, b \mapsto b, c \mapsto b$. If $ab^n a$ occurs in Y, then n must be even. $Y = X_{\mathcal{L}}$ where

$$\mathcal{L} = \{ab^{2n+1}a : n \in \mathbb{N}\} \ .$$

  Y (the "even system") is sofic and not SFT.

II. Overview: Z^d SFTs and sofic shifts, $d = 1$ vs. $d \geq 2$

For Z SFTs:

- There are computable, fine invariants.

- Invariants and structure have an algebraic quality.
  (Any SFT is topologically isomorphic to X_B for some matrix B. Various algebraic invariants of B give invariants of topological isomorphism for X_B.)

- Qualitatively, mixing Z SFTs have a homogeneous structure, rich in subsystems and quotients.

- Generally problems of SFTs reduce easily to problems of mixing SFTs.

For Z^d SFTs,

- Essentially nothing can be computed for a general arbitrary Z^d SFT if $d \geq 2$.
(Berger, Robinson, Kari, ...)

- The landscape of general possibilities is recursion-theoretic.
Hochman-Meyerovitch and Hochman give constructive results and techniques of generality unprecedented in this area.

- SFTs, including mixing SFTs, are qualitatively heterogeneous.

- SFT problems do not reduce to problems of mixing SFTs.

III. Specific Contrasts

Emptiness

Let $\mathcal{F}$ be a finite set of excluded configurations defining a Z^d SFT $\sigma_{\mathcal{F}}$.

For $d = 1$: easily compute a matrix B such that $(X_{\mathcal{F}}, \sigma_{\mathcal{F}}) \cong (X_B, \sigma_B)$. Now X_B is empty iff B is nilpotent.

For each $d \geq 2$: no Turing machine can take arbitrary input $\mathcal{F}$ and decide if $X_{\mathcal{F}}$ is empty. (Berger, 1960's)

Entropy

Given a Z^d subshift $(X, \sigma|X)$:
let $B_n = \{v \in Z^d : 0 \leq v_i \leq n-1, 1 \leq i \leq d\}$.
The entropy of this shift (as a Z^d action) is

$$h(\sigma|X) = \lim_n \frac{1}{n^d} \log \operatorname{card}\{x|_{B_n} : x \in X\} \ .$$

This is the main numerical measure of the complexity of a subshift.

For $X = \{0, \ldots, N-1\}^{Z^d}$, the entropy is simply $\log N$.

For $X = X_B$, the entropy is the log of the spectral radius of the matrix B.

For each $d \geq 2$: there are not many exact computations of entropy for systems "in nature"; and there is no Turing machine which can compute $h(\sigma_{\mathcal{F}})$ for an arbitrary finite collection of configurations $\mathcal{F}$.

TFAE (Lind) for a nonnegative real number α.

- α is the entropy of some Z SFT
- $\alpha = (1/n)\log\lambda$ where λ is a positive algebraic integer such that $\lambda > \lambda_i$ for every other algebraic conjugate λ_i of λ.

For each $d \geq 2$: TFAE (Hochman-Meyerovitch) for a nonnegative real number α.

- α is the entropy of some Z^d SFT
- $\alpha = \lim_n \alpha_n$, where $(\alpha_n)_{n\in\mathbb{N}}$ is a decreasing recursive sequence of rational numbers.

A recursive sequence is the output of a Turing machine – intuitively, a sequence produced by any kind of algorithm you could imagine a computer implementing.

Periodic points

For a Z SFT (X_B, σ_B):

- the number of fixed points of σ_B^n is trace(B^n).
- The polynomial det$(I - tB)$ encodes all periodic point count information for the SFT.
- A nonempty SFT always contains periodic periodic points; and these are dense in the recurrent (important) part of the dynamics, and are an important tool for analysis.

For a Z^d SFT, $d \geq 2$:

- It can have no periodic points. (Berger, Robinson)

- There is no decision procedure for determining if it has a periodic point

- We have no finite encoding of the periodic point structure.

Effective Systems

Hochman has defined an Effective Symbolic System (ESS) to be a Z^d subshift $\sigma_{\mathcal{L}}$ such that the defining set $\mathcal{L}$ of forbidden finite configurations is the output of a Turing machine.

THEOREM (Hochman) For each $d \geq 3$, up to topological conjugacy the following classes of Z subshift are the same:

- Z subshifts isomorphic to σ^{e_1} for some Z^d sofic shift σ

- The class of ESS's.

(With SFT in place of sofic, the class of ESS's becomes just slightly more narrow.)

Heuristically: not only are we faced with many bad examples: in fact every (bad) thing we could possibly imagine happening, does happen.

What if σ^{e_1} is not isomorphic to a subshift?

Hochman defined a Z^d Effective Dynamical System (EDS) in a similar fashion to be a Z^d action on a zero dimensional space describable by a Turing machine. He showed

- For σ a Z^d SFT, σ^{e_1} is an Z EDS

- For any Z EDS T, there is a Z^d SFT σ such that σ^{e_1} is "almost" the same as T (there is an odometer U such that σ^{e_1} is an extension of $T \times U$ by a map one-to-one a.e. w.r.t. all invariant Borel probabilities).

An application to cellular automata

Recall, a d-dimensional cellular automaton map (on N symbols) is a block code from $\mathcal{A}^{Z^d}$ to itself.

A trick shows the following classes are essentially the same dynamically:

- d-dimensional cellular automaton maps

- maps σ^{e_1} coming from Z^d SFTs σ

THEOREM (Hochman) Suppose $d \geq 3$. TFAE for a nonnegative real number α.

- α is the entropy of an EDS.
- α is the entropy of a d-dimensional c.a. map.
- α is the liminf of a sequence of rational numbers produced by a Turing machine. (!)

Mixing

- A Z^d subshift (X, σ) is *mixing* if any two legal finite configurations can occur at all but finitely many separations.

- That is, for all x, y in X, for all n we have for all but finitely many $u \in Z^d$ there exists $w \in X$ such that
$w|_{B_n} = x|_{B_n}$ and $w|_{u+B_n} = y|_{B_n}$.

- Generally problems of Z SFTs reduce easily to problems of mixing Z SFTs.

- For $d \geq 2$, the mixing condition splinters into a host of different conditions.

Subsystems

Nontrivial mixing Z SFTs have a homogeneous structure, rich in subsystems and quotients:

- Krieger Embedding Theorem $\Longrightarrow$ if (X, σ_X) is a mixing Z SFT and (Y, σ_Y) is a Z subshift with no periodic points and $h(\sigma_Y) < h(\sigma_X)$, then (Y, σ_Y) is topologically conjugate to a subshift contained in (X, σ_X).

- Jewett-Krieger Theorem: every finite entropy measurable Z-system is realized by a uniquely ergodic Z-subshift

- So: a nontrivial mixing Z SFT (X, σ_X) contains a vast family of pairwise disjoint minimal subsystems with entropy close to $h(\sigma_X)$.

- Given a proper subsystem of a mixing Z SFT, all the embeddings above can be chosen with images missing that subsystem. (There is always room at the Krieger Hotel.)

A mixing Z sofic shift X contains an increasing union of mixing Z SFTs X_n with $\lim_n h(X_n) = h(X)$. So the theorem above holds with sofic in place of SFT.

Z SFTs and Z sofic shifts also enjoy a rich supply of quotient maps.

THEOREM

1. If X is a Z sofic shift, and Y is a mixing Z SFT with $h(X) > h(Y)$, then Y is a quotient of X.

2. If Y is a Z SFT and $h(Y) \geq \log N$, then Y has as a factor the full shift on N symbols.

Johnson and Madden asked whether (2) generalizes to Z^d SFTs. Their work, as extended by Desai, proved the conclusion of (2) for a Z^d SFT Y with the "corner gluing" mixing condition when $h(Y) > \log N$.

THEOREM (B-Pavlov-Schraudner) Given $d \geq 2$ and $M \in \mathbb{R}$, there is a Z^d sofic shift σ_X with the following properties.

- $h(\sigma_X) > M$ and σ_X is mixing

- X contains a unique minimal subsystem, which is a fixed point for σ_X.

- Any subshift quotient Y of X satisfies the following:

  – If Y is SFT, then Y is a fixed point.

  – Y is not block-gluing (mixing on block shapes with uniform separation).

  – Y supports no σ-invariant measure which is of completely positive entropy.

  – Y does have a subshift factor of topologically completely positive entropy.

THEOREM Given $d \geq 2$ and $M \in \mathbb{R}$, there is a Z^d SFT σ_X with the following properties.

- $h(\sigma_X) > M$
- X contains an SFT W such that $h(W) = 0$ and every minimal subsystem of X is contained in W.
- Any subshift quotient Y of X satisfies the following:
  - Y has a sofic zero entropy subsystem containing all minimal subshifts of Y.
  - Y is not, for example, a full shift on two symbols.
  - Y is not block-gluing (mixing on block shapes with uniform separation).

- Furthermore, for a subshift quotient Y of X,
  - Y supports no σ-invariant measure which is of completely positive entropy.
  - Y does have a subshift factor of topologically completely positive entropy.
- In the case $d = 2$, the Z^d SFT σ_X can be chosen mixing and of topologically completely positive entropy.

IV. A Hint of Proof?

We can indicate how Hochman's subdynamics theorem lets one easily construct a nonmixing sofic example Y for $d = 3$.

- Construct an effective Z subshift W such that arbitrarily large blocks of 0's occur syndetically in all points, and every W word occurs with positive frequency in every point.

- By Hochman: for $i = 1, 2, 3$, pick a Z^3 sofic shift T_i for which σ^{e_i} is a copy of W.

- Then in each coordinate of $T_1 \times T_2 \times T_3$, for each $M \in \mathbb{N}$, strings of M consecutive zeros occur syndetically.

- Let W be the quotient of $T_1 \times T_2 \times T_3$ by the map which replaces a symbol (a, b, c) with 0 if any of a, b, c is zero, and otherwise replaces (a, b, c) with 1.

- For every M and $w \in W$, every finite configuration in W occurs inside some large block configuration on which the boundary is covered by M-thick slabs of zeros.

- Define Y by freely allowing the replacement of 1 in a configuration with symbols from $\{1, 2, ..., k\}$ for k large.

- Easily:

  - $h(\sigma_X) > M$ (for large enough k)

  - σ_X has a unique minimal subsystem, 0^{Z^3}.

  - The only SFT which is in X or is a quotient of X is a fixed point.