Expansive maps commuting with a shift of finite type

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We will consider maps which commute with a given shift of finite type (SFT).

We have two themes.

1. Which SFTs can commute with a given SFT?

2. If φ is an expansive homeomorphism which commutes with an irreducible SFT, must φ be itself SFT? [Nasu]

Question 1 is relevant to providing context for the intense study of invariant measures for commuting algebraic maps: how much does the phenomenon of few jointly invariant measures generalize?

How a matrix defines an SFT

Given a square matrix A with entries in $\mathbb{Z}_+$:

- view A as the adjacency matrix of a directed graph $\mathcal{G}^A$

- $A(i,j) =$ the number of edges from vertex i to vertex j

- Σ_A is the set of bisequences $x = \dots x_{-1}x_0x_1\dots$ corresponding to walks through $\mathcal{G}_A$: each x_i is an edge, and for all i the terminal vertex of x_i equals the initial vertex of x_{i+1}.

- The shift map $\sigma_A : \Sigma_A \to \Sigma_A$ is the SFT defined by A. Here $\sigma_A : x \mapsto y$, where $y_i = x_{i+1}$ for all i. Σ_A is compact metrizable and σ_A is a homeomorphism.

Definitions

- An *automorphism* of a continuous map f is a homeomorphism U commuting with f ($Uf = fU$).

- Continuous maps f, F are *topologically conjugate* ($f \sim F$) if there exists a homeomorphism h such that h $Fh = hf$.

- Continuous maps f, g *can commute* if $\exists F \sim f, G \sim g$ with $FG = GF$ (i.e., isomorphic copies of f and g *do* commute).

- S is SFT if $S \sim \sigma_A$, for some A.

Which SFTs can commute?

Periodic point obstructions

If bijections S, T commute, then for each n, S maps points of T-period n to other points of T-period n. If T has only k points of period n, then these are periodic points of S, of S-period at most k.

Low-order periodic point conditions of this sort sometimes imply two maps cannot commute.

For example, if $|\text{Fix}(S)| = 1$ and $|\text{Fix}(T)| = 0$, then S and T cannot commute, because the fixed point set of S (one point) would have to be fixed by T – but T has no fixed point.

For another example: let us see that $\sigma_{[2]}$ and $\sigma_{[3]}$ (the full shifts on two and three symbols) cannot commute.

Suppose $S \sim \sigma_{[2]}$, $T \sim \sigma_{[3]}$ and $ST = TS$.

- Claim: the two fixed points of S are a T-orbit of size 2.
  Proof: S has two fixed points. Either they are fixed points of T, or they are a T-orbit of size 2. If they are fixed by T, then there is just one more fixed point of T, which must then be fixed by S, so S has a third fixed point. This is a contradiction.

- Claim: the single S-orbit of size 2 is a T-orbit of size 2.
  Proof: the same.

- Now we know the points of S period 1 and 2 form two T-orbits of size 2. There are exactly three T-orbits of size 2. The remaining orbit is then an S-invariant set of two points, i.e. it contains points of S-period 1 or 2. This is a contradiction. QED.

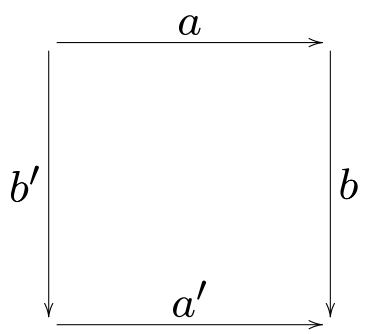
Commuting SFTs from commuting matrices

Note: $AB = BA$ does not guarantee that σ_A, σ_B can commute. (E.g. $[A] = 2, B = [3]$). However:

Proposition. Suppose A, B are commuting $\mathbb{Z}_+$ matrices. Then there are homeomorphisms S, T such that $ST = TS$ and $S^iT^j \sim \sigma_{A^iB^j}$ for $i, j > 0$.

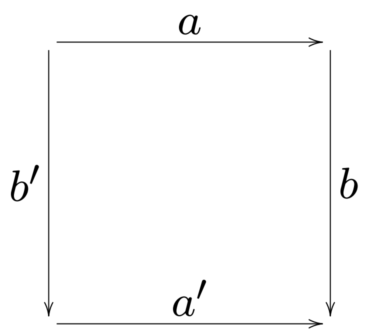
Let us see a simple construction for this, which comes essentially from the 1995 AMS Memoir "Textile Systems" of M. Nasu.

Suppose A and B are $n \times n$ matrices over $\mathbb{Z}_+$, with $AB = BA$. View A and B as adjacency matrices for two directed graphs, with disjoint edge sets and a common vertex set $\{1, 2, \ldots n\}$. Say e.g. an ab path from i to j is an A edge from i to some k followed by a B edge from that k to j. "$AB = BA$" means that for each pair i, j the number of ab paths from i to j equals the number of ba paths from i to j. Thus we can build a set $\mathcal{W}$ of Wang tiles

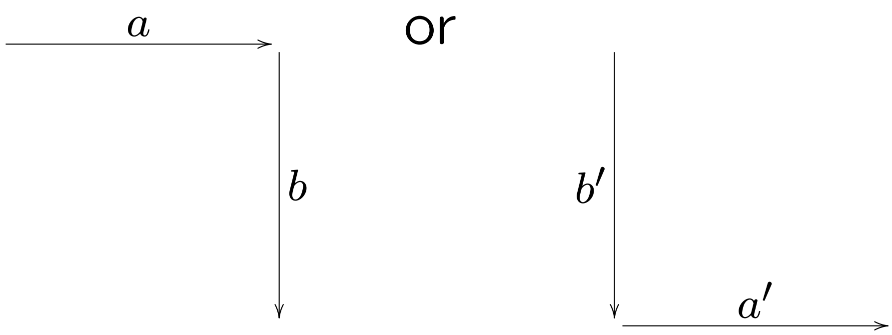


such that each ab path is the top/right of exactly one tile and each ba path is the left/bottom of exactly one tile. (In the tile pictured, a, a' are A-edges and b, b' are B-edges.)

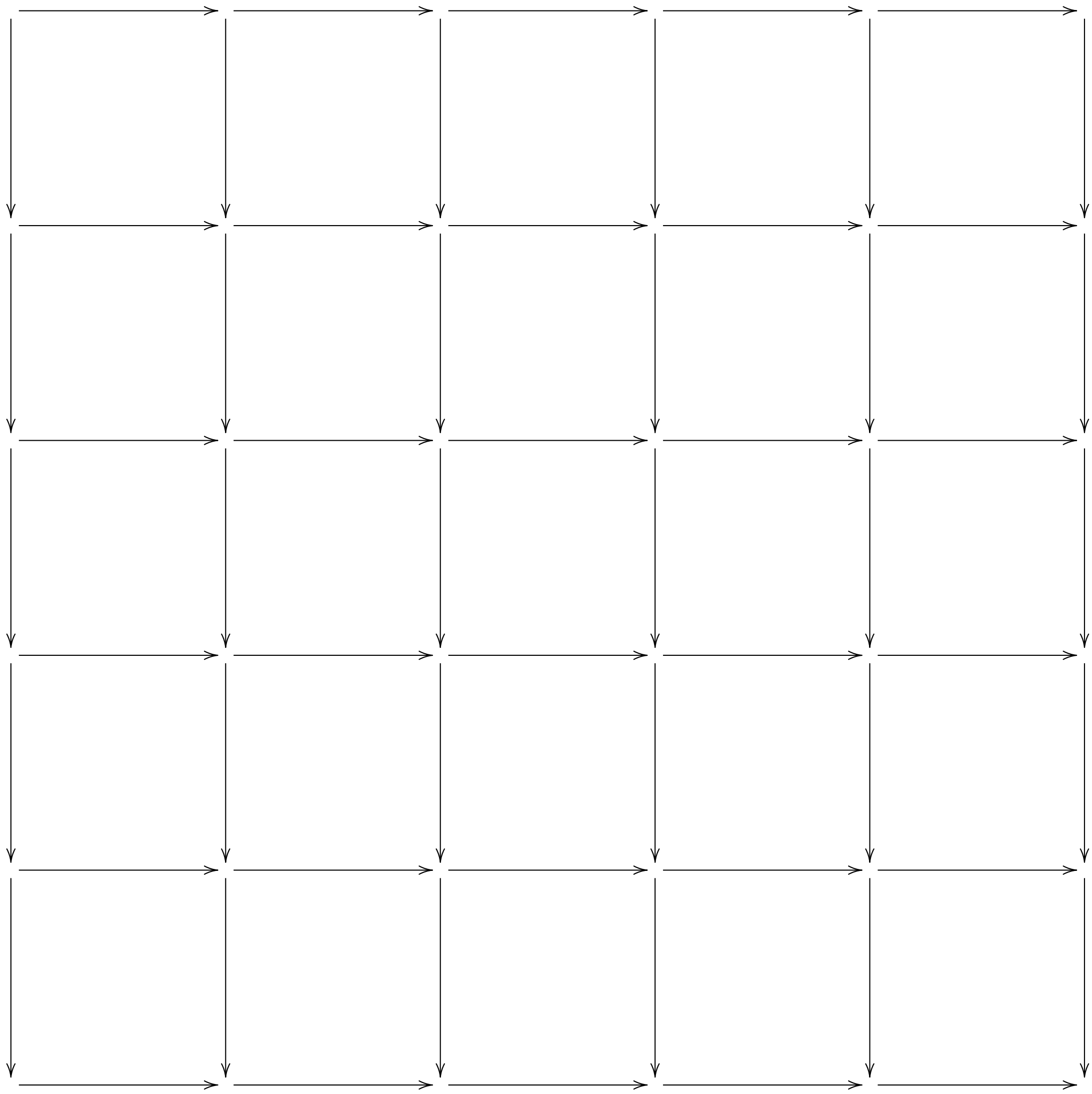
Thus each Wang tile



is determined by either of the paths

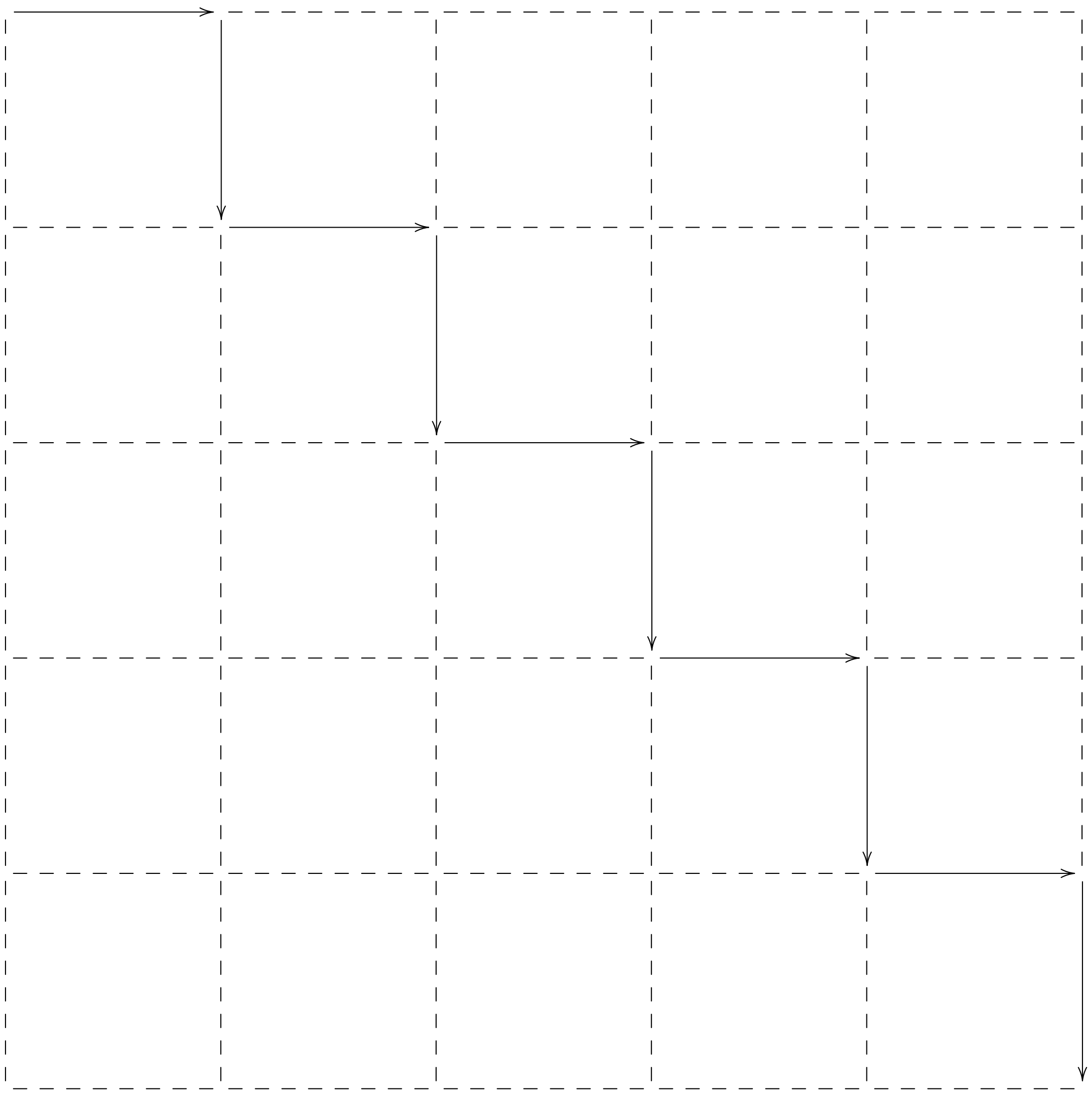


Let the tile sides be unit length and let W be the space of infinite Wang tilings of the plane with $\mathcal{W}$, with tile corners on $\mathbb{Z}^2$. A tiling is a Wang tiling iff labels on adjoining tile sides match. Here the labels are edges.

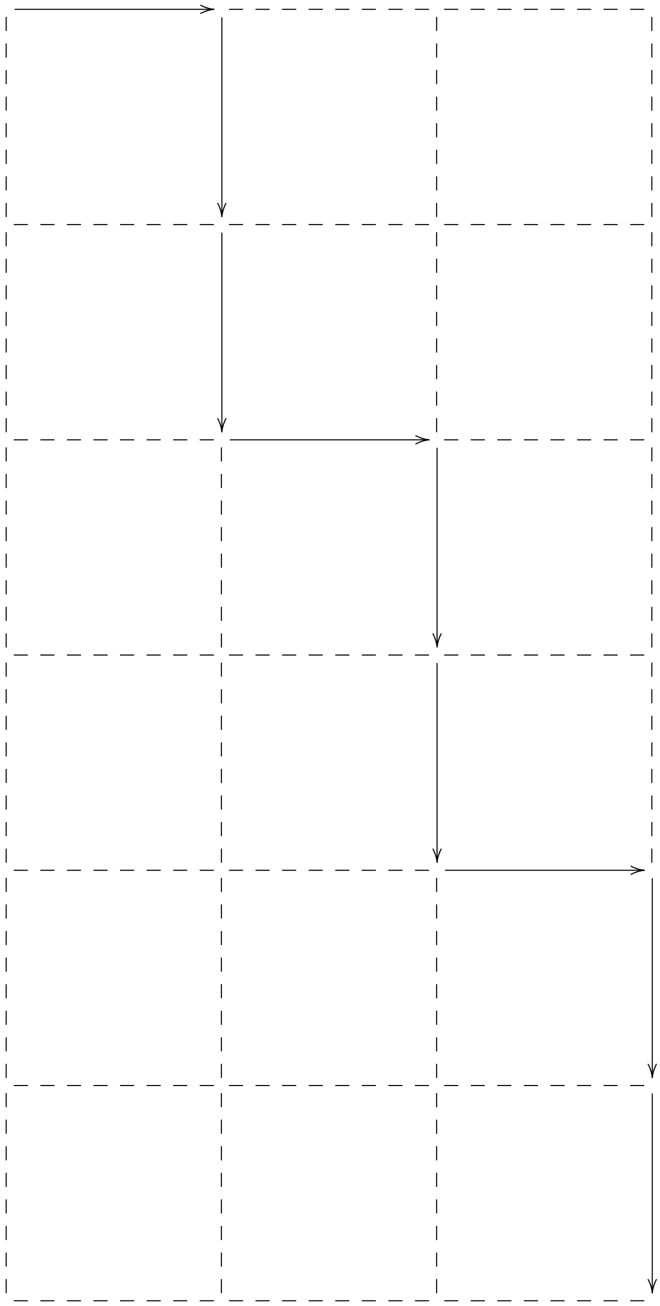


E.g., above is a finite piece of a point in W, with edge-name labels suppressed. For $\mathbf{v} \in \mathbb{Z}^2$, let $\alpha_{\mathbf{v}}$ denote the shift map on W in direction $\mathbf{v}$.

The bijections cited two slides back show solid lines below determine the dashed lines. Thus $\alpha_{(1,-1)}$ is expansive and conjugate to the SFT σ_{AB}.



Likewise the solid squares below determine the rest, and $\alpha_{(1,-2)} \sim \sigma_{AB^2}$.



Given the commuting matrices A, B we showed how to embed $\sigma_{A^i B^j}$ into a commuting family of maps when $(i, j) = (1, 1)$ or $(i, j) = (1, 2)$. The argument is the same for $i > 0, j > 0$.

The construction works also for onesided SFTs. The onesided SFTs are defined from matrices just like the usual SFTs, except, the domain is a space of onesided sequences

$$x_0 x_1 \cdots$$

rather than twosided sequences.

$$\cdots x_{-1} x_0 x_1 \cdots$$

A onesided SFT is not injective but it is a local homeomorphism.

We know a lot about onesided SFTs which can commute:

- Commuting onesided SFTs can be presented by commuting matrices over $\mathbb{Z}_+$ [Nasu]

- There is a refined dimension group approach putting drastic constraints on which onesided SFTs can commute [B-Fiebig-Fiebig, generalizing Blanchard-Maass]

- Commuting onesided SFTs have the same measure of maximal entropy [B-Fiebig-Fiebig]

All of this is NOT TRUE in general for twosided SFTs.

Dimension groups

For a $k \times k$ $\mathbb{Z}_+$-matrix A, set $G_A = \varinjlim_A \mathbb{Z}^k$.

Examples:

- for $A = [2]$, $G_A \cong \mathbb{Z}[1/2]$
- for $A = [k]$, $G_A \cong \mathbb{Z}[1/k]$
- for A $n \times n$ with $|\det(A)| = 1$, $G_A \cong \mathbb{Z}^n$
- In general G_A is isomorphic to a subgroup of a finite dimensional rational vector space.

With an extra order structure, G_A becomes a dimension group. These groups play a big role in the theory of SFTs. There is also a natural action $\widehat{A}$ of A on G_A.

For mixing SFTs σ_A and σ_B with $AB = BA$, the groups G_A and G_B are the same, with $\widehat{A}\widehat{B} = \widehat{B}\widehat{A}$. The group and action are invariants of the topological conjugacy class of an SFT. So, if $AB = BA$ and the mixing SFTs σ_A and σ_B can commute, then there must be an isomorphism $G_A \to G_B$ which carries $\widehat{A}$ to an action commuting with $\widehat{B}$. In particular, for large n the matrices A^n and B^n have equal rank.

There is a dramatic example of Nasu which shows not all commuting SFTs can arise from commuting matrices.

EXAMPLE (Nasu 95): $\sigma_A T = T\sigma_A$, $T \sim \sigma_B$,

- $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

- $\chi_B(x) = (x+1)^2(x^3 - 2x^2 + x + 1)$.

(σ_A and T do not even have the same measure of maximal entropy, because the range of the two measures of maximal entropy on clopen sets is not the same).

Nasu gave a complicated algorithm which, given an automorphism U of an irreducible SFT, will find a matrix B such that $\sigma_B \sim U$, *IF* U is SFT. The example above came from applying the algorithm to a particular automorphism.

Nasu's example shows that SFTs σ_A and σ_B can commute without the matrices A, B being algebraically related in any obvious way.

CONJECTURE: Suppose S and T are mixing SFTs. Then for all large i, j, S^i and T^j can commute.

There is no set theoretic periodic point obstruction to the conjecture (exercise).

Note, if mixing SFTs S and T commute, then each fixes the measure of maximal entropy of the other. This begins an exploration of multiple jointly invariant measures.

Let us verify the conjecture in a special case.

Proposition. Suppose σ_A, σ_B are mixing SFTs, and there is an isomorphism $G_B \to G_A$ which carries the action of B to an automorphism of G_A commuting with the action of A. Then for all large k, the SFT $(\sigma_B)^k$ can commute with σ_A.

Proof outline. We can find a matrix C commuting with A such that (i) the action of C on G_A is isomorphic to that of B on G_B, (ii) $AC = CA$ and (iii) for large k, C^k is positive.

Then for large k, σ_{C^k} commutes with σ_A, and σ_{C^k} is topologically conjugate to $(\sigma_B)^k$. QED

Theme 2:

Question (Nasu 1989). Must an expansive automorphism of an irreducible SFT be itself SFT?

This question is still (!) open for twosided SFTs.

For onesided SFTs: the answer is Yes [Kurka, Nasu]. Moreover:

- Theorem: An expansive automorphism of a onesided full shift must be SFT [Nasu] and it must be shift equivalent to a full shift [B-Maass].

- Conjecture [B-Maass] An expansive automorphism of a onesided full shift must be topologically conjugate to a full shift.

We also know just what the corresponding full shifts have to be: for a onesided full shift on n symbols, the full shifts on k symbols corresponding to the automorphism are those for which k has the following properties ([Nasu] following [B-Maass]):

- A prime p divides n iff it divides k
- If a prime p divides n, then p^2 divides k.

In contrast we still know very little in the case of twosided case. Let us see what we know.

EXAMPLE (D. Fiebig, 1996) A reducible SFT S with an expansive automorphism U which is not SFT.

The example is utterly simple.
S consists of two fixed points p, q and two connecting orbits from p to q. Concretely, the fixed points and connecting orbits are

$$p = \dots 000 \dots \quad , \qquad \dots 0002111 \dots \quad ,$$
$$q = \dots 111 \dots \quad , \qquad \dots 0003111 \dots \quad .$$

$U = S$, except that $U = S^{-1}$ on one of the connecting orbits. U is expansive and totally chain transitive but not SFT. D.F. (easily) also elaborated this example to positive entropy.

We have a result which at least addresses a meaningful case of our question.

THEOREM (B. 2004) A strictly sofic AFT (almost finite type) shift S cannot commute with a mixing SFT T.

A sofic shift is a subshift which is a quotient/factor of an SFT. Above, “mixing” can be replaced by “chain recurrent” (which for sofic means dense periodic points). A sofic shift is AFT if it a factor of an irreducible SFT by a map which is biclosing (injective on stable sets and on unstable sets). In this case there is a canonical such map which is one-to-one almost everwhere. The AFT sofic shifts enjoy various properties and seem to be the one big, natural class of nice sofic shifts.

We’ll outline the proof of the theorem (to give an idea of methods). Let S and T be as above. Notation: a map f is *totally P* if f^n has property P for all $n > 0$.

Step 1: LEMMA A.

An expansive automorphism of a mixing SFT is totally chain transitive.

(A subshift is totally chain transitive iff for all $n > 0$ the SFT built from its allowed words of length n is mixing.)

Step 2: setup with canonical cover.

We suppose S is strictly sofic AFT and $ST = TS$. Lemma A then implies S is mixing. Let $\pi : \widetilde{S} \to S$ be the canonical biclosing cover of S by a mixing SFT. By "canonical", T lifts to an automorphism $\widetilde{T}$ of $\widetilde{S}$.

Step 3: $\widetilde{T}$ is expansive.

Fibers of π are uniformly separated, because T is expansive. Points within fibers are uniformly separated because π is biclosing.

STEP 4: $\widetilde{T}$ is a mixing SFT.
Because $\widetilde{T}$ is an expansive automorphism of the mixing SFT $\widetilde{S}$, Lemma A implies that $\widetilde{T}$ is totally chain transitive. Now $\pi : \widetilde{T} \to T$ is a closing factor map from a totally chain transitive subshift onto a mixing SFT. A sofic argument of Kitchens adapts to this situation to show $\widetilde{T}$ must be a mixing SFT.

From the sofic automorphism S of the mixing SFT T, we lifted to the mixing SFT automorphism $\tilde{S}$ of the mixing SFT $\tilde{T}$. Por que?

STEP 5: the contradiction.
Now $\pi : \widetilde{T} \to T$ is a biclosing map of mixing SFTs, hence constant-to-one. But $\pi : \widetilde{S} \to S$ is 1-1 a.e. (as the canonical cover) but not everywhere (since S is strictly sofic). QED

The heart of the proof of Lemma A is the following lemma (surprisingly difficult ?).

Lemma. Suppose T is mixing SFT; S is an expansive automorphism of T; B is a closed open set; and $SB = B$. Then B is trivial.

Proof sketch. Let μ be the measure of max. entropy for T; it is S-invariant. Choose k large enough that $S' = S^kT$ and S lie in the same expansive component of the $\mathbb{Z}^2$ action, and therefore have the same Pinsker algebra with respect to μ, by a directional coding argument from B-Lind [Expansive subdynamics].

Now suppose the partition $\mathcal{B} = \{B, B'\}$ is non-trivial. Then the following list gives us the desired contradiction.

- $h(S', \mu, \mathcal{B}) = 0$.
  This holds because $h(S, \mu, \mathcal{B}) = 0$ (since $SB = B$) and S and S' have the same Pinsker algebra w.r.t. μ.

- $h(S', \mu, \mathcal{B}) = h(T, \mu, \mathcal{B})$.
  This holds since $SB = B$ implies for $n > 0$ that

$$\bigvee_{i=0}^{n} (S')^i B = \bigvee_{i=0}^{n} (TS^k)^i B = \bigvee_{i=0}^{n} T^i B \ .$$

- $h(T, \mu, \mathcal{B}) > 0$.
  This holds since μ is a K-automorphism.

QED

Summary

This has been a talk more about questions than answers.

1. Which SFTs can commute?

We have a pretty good understanding of how SFTs commute in an algebraically related way (which covers the onesided SFT case).

It would be very interesting to have any general construction of commuting SFTs which are not algebraically related. This may be "simply" a matter of more sophisticated combinatorics on Wang tilings.

2. Must an expansive homeomorphism commuting with an irreducible SFT be itself SFT?

For onesided SFTs: yes (Kurka, Nasu).

For twosided SFTs: this is still very open.