A Direct Method For Designing FUNTFs

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Background

Matrix Products for Column Vectors in $\mathbb{R}^d$

- The dot product of $\overline{x}, \overline{y}$ in $\mathbb{R}^d$ is $\overline{x}^T\overline{y} \in \mathbb{R}$. The length of $\overline{x}$ satisfies $\|\overline{x}\| = \sqrt{\overline{x}^T\overline{x}}$.
- The tensor product is the $d \times d$ matrix $\overline{x}\,\overline{y}^T$. The tensor product $\overline{x}\,\overline{x}^T$ is symmetric, and positive semi-definite since

$$\overline{y}^T\left(\overline{x}\,\overline{x}^T\right)\overline{y} = \left(\overline{x}^T\overline{y}\right)^2 \geq 0.$$

Frame Theory

- A FUNTF is a **F**inite **U**nit-**N**orm **T**ight **F**rame.
- A real finite frame is a collection of vectors $\{\overline{x}_i\}_{i=1}^N$ in $\mathbb{R}^d$ for which there exists constants (the frame bounds) $0 < A \leq B$ satisfying

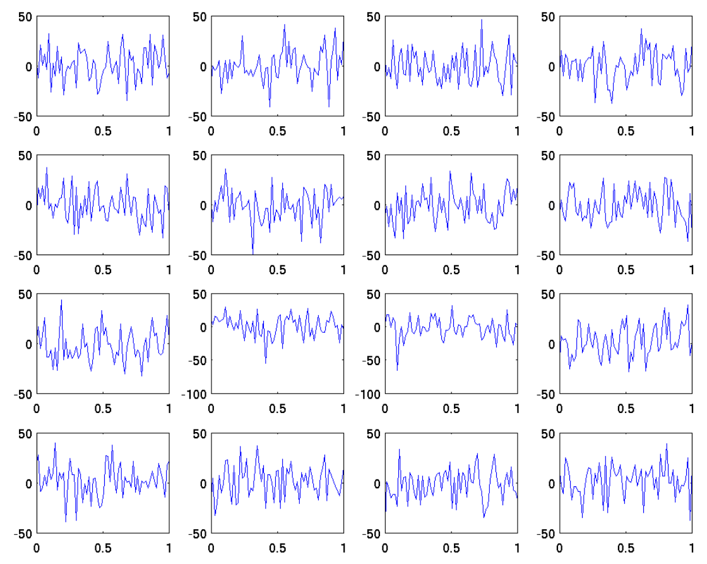
$$A\|\overline{y}\|^2 \leq \sum_{i=1}^{N}\overline{y}^T\left(\overline{x}_i\overline{x}_i^T\right)\overline{y} \leq B\|\overline{y}\|^2 \text{ for all } \overline{y} \in \mathbb{R}^d.$$

- If $A = B$, then the frame is called tight. If $\|\overline{x}_i\| = 1$ for all $i = 1, \ldots, N$, then the frame is called unit-norm.
- The frame operator is $S = \sum_1^N \overline{x}_i\overline{x}_i^T$, so $A\|\overline{y}\|^2 \leq \overline{y}^T S\,\overline{y} \leq B\|\overline{y}\|^2$ for all $y \in \mathbb{R}^d$.
- The frame operator of a FUNTF is a multiple of the $d \times d$ identity matrix: $\frac{N}{d}I_{d\times d}$. It is also positive definite, so we write $0 \prec S$.

Why Design FUNTFs?

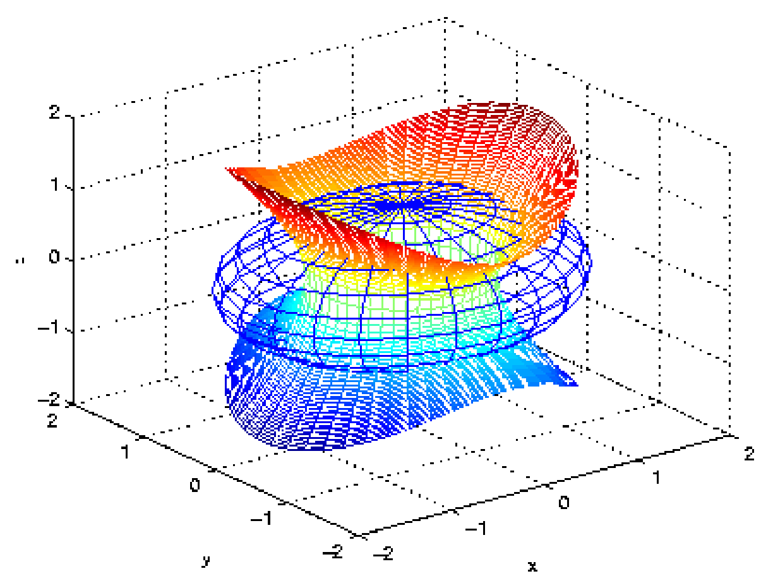
Synchronous CDMA

- **C**ode **D**ivision **M**ultiple **A**ccess
- Used for cell phones
- Need Optimal Codes

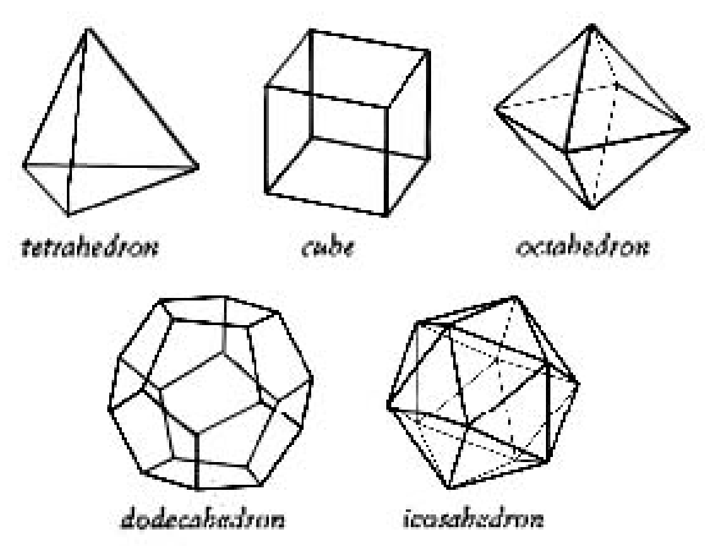


Beautiful Mathematics

- Kadison-Singer conjecture
- Algebraic geometry



Examples of FUNTFs



The Constraints for Frame Design

Suppose $A \in M_d(\mathbb{R})$ is symmetric and positive definite ($0 \prec A$) and let $\overline{x} \in \mathbb{R}^d$. Then

$$0 \preceq A - \overline{x}\,\overline{x}^T \iff \overline{x}^T A^{-1}\,\overline{x} \leq 1$$

and

$$rank(A - \overline{x}\overline{x}^T) < rank(A) \iff \overline{x}^T A^{-1}\overline{x} = 1$$

Proof.

Note that $A - \overline{x}\,\overline{x}^T$ has at most one non-negative eigenvalue by the interlacing inequalities for eigenvalues. Now, the determinant of $A - \overline{x}\,\overline{x}^T$ is the product of the eigenvalues, so we infer that this difference will be positive semi-definite if and only if the determinant is non-negative.

Since A is positive definite, it is invertible, and we may use the Sherman-Morrison determinant formula:

$$\det\left(A - \overline{x}\,\overline{x}^T\right) = \det(A)\left(1 - \overline{x}^T A\overline{x}\right).$$

But $\det(A) > 0$ since A is positve definite, so the left term is non-negative if and only if $0 \leq 1 - \overline{x}^T A\overline{x}$. Thus, $A - \overline{x}\,\overline{x}^T$ is positive semi-definite if and only if $\overline{x}^T A^{-1}\,\overline{x} \leq 1$. The rank inequality follows from similar considerations. □

FUNTF Design Algorithm

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(0) Input $N, d$. Set $S = (N/d)I_{d\times d}$.
(1) For $i = \{1, \ldots, N-d\}$
    ▸ Choose $\overline{x}_i$: $\overline{x}_i^T S^{-1}\overline{x}_i < 1$.
    ▸ $S \leftarrow S - \overline{x}_i^T\overline{x}_i$.
(2) For $i = \{N-d+1, \ldots, N\}$
    ▸ Set $\widetilde{S}^{-1}\overline{y} = \overline{v}$: $\overline{v} \in \text{Image}(S)$, $S\overline{v} = \overline{y}$.
    ▸ Choose $\overline{x}_i$: $\overline{x}_i \in \text{Image}(S)$, $\overline{x}_i^T\widetilde{S}^{-1}\overline{x}_i = 1$.
    ▸ $S \leftarrow S - \overline{x}_i^T\overline{x}_i$.
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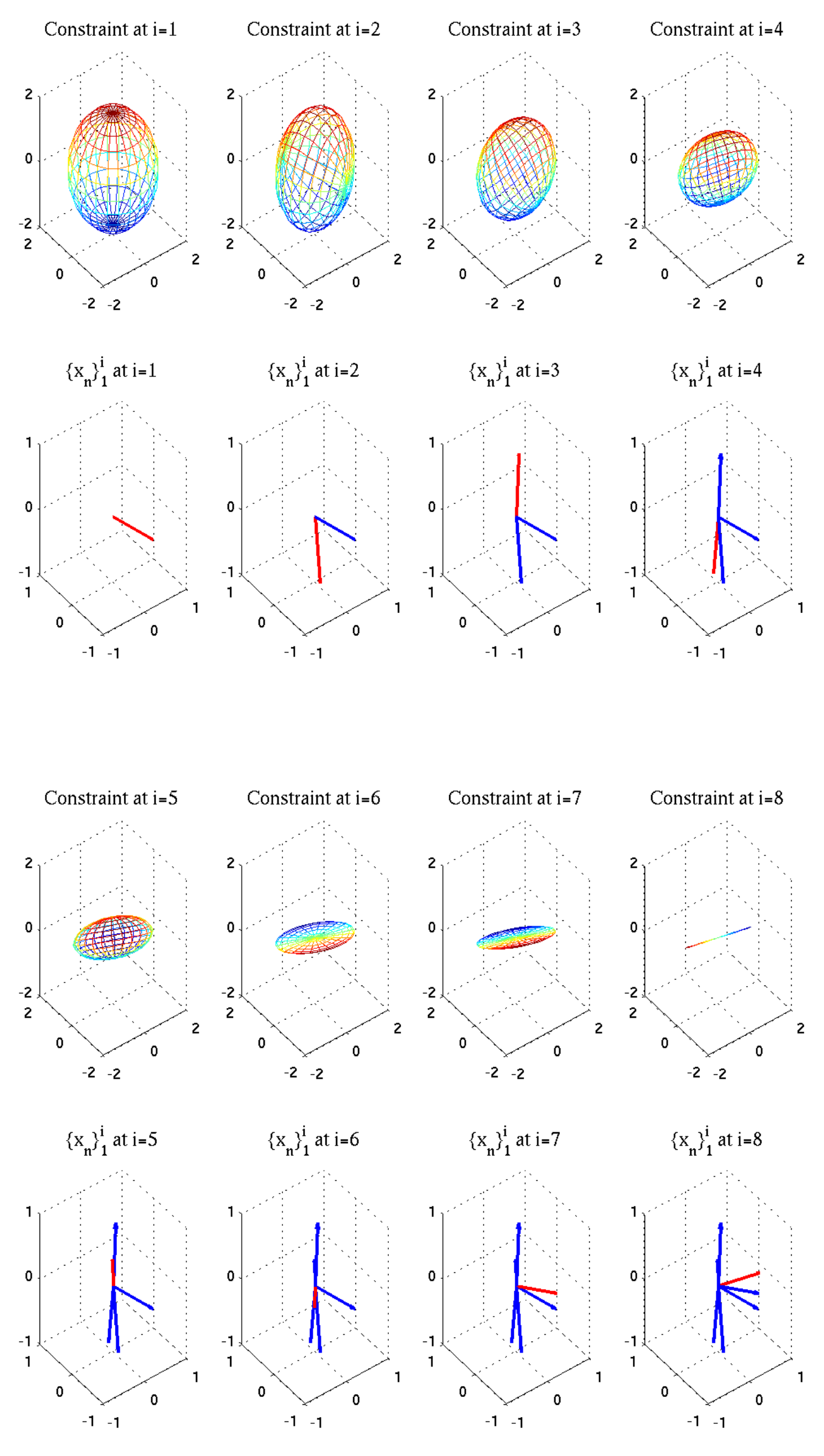
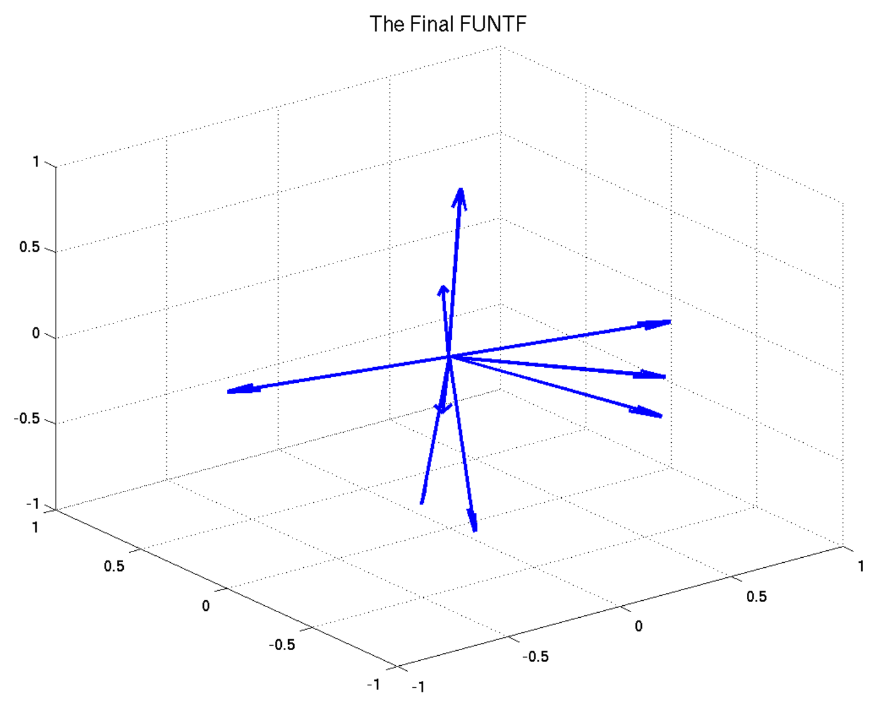
Example: N=9, d=3





The Sherman-Morrison Determinant Formula

$$\det\left(A - \overline{x}\,\overline{x}^T\right) = \det(A)\cdot\left(1 - \overline{x}^T A^{-1}\overline{x}\right)$$

Proof.

Note that A must be invertible to apply this formula. Consider the case $d = 2$ (this proof is easily adapted to general d). Setting $A = [\overline{a}_1\,\overline{a}_2]$ and $\overline{x} = [x_1\,x_2]^T$, we have that

$$\begin{aligned}
\det\left(A - \overline{x}\overline{x}^T\right) &= \det\left([\overline{a}_1 - x_1\overline{x}\ \ \overline{a}_2 - x_2\overline{x}]\right) \\
&= \det\left([\overline{a}_1\,\overline{a}_2]\right) - \det\left([x_1\overline{x}\,\overline{a}_2]\right) && (1) \\
&\quad -\det\left([\overline{a}_1\,x_2\overline{x}]\right) + \det\left([x_1\overline{x}\,x_2\overline{x}]\right) \\
&= \det(A) - x_1\det(A_1(\overline{x})) - x_2\det(A_2(\overline{x})) + 0 && (2) \\
&= \det(A) - \det(A)x_1\left(A^{-1}\overline{x}\right)_1 - \det(A)x_2\left(A^{-1}\overline{x}\right)_2 && (3) \\
&= \det(A) - \det(A)\left(\overline{x}^T A^{-1}\overline{x}\right) \\
&= \det(A)\left(1 - \overline{x}^T A^{-1}\overline{x}\right).
\end{aligned}$$

To get to line (1), we used the fact that the determinant is a multi-linear function of the columns. To get to line (2), we note that the determinant of a matrix with linearly dependent columns is zero. To get to line (3), we use Cramer's rule. □

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