STAT 100 Lecture 15 : Confidence Interval for a Population Mean

Nate Strawn

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Last Time...

1. Statistical Inference
2. Point estimation of a population mean

Today's Agenda

1. A confidence interval when normal population and σ known.
2. Interpretation of confidence intervals.
3. Large sample confidence interval for μ.
4. Confidence interval for a parameter.

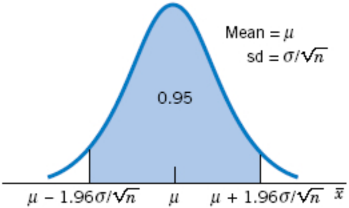
A confidence interval when normal population and σ known.

Assume that

- the population is Normal
- the standard deviation σ is known.
- $(X_1, X_2, \ldots, X_n)$ is a random sample from the population. It means that $X_1, X_2, \ldots, X_n$ are independent.

Then

$$P[\mu - 1.96 \times \sigma/\sqrt{n} < \overline{X} < \mu + 1.96 \times \sigma/\sqrt{n}] = 0.95$$



A confidence interval when normal population and σ known.

$$P[\mu - 1.96 \times \sigma/\sqrt{n} < \overline{X} < \mu + 1.96 \times \sigma/\sqrt{n}] = 0.95$$

is equivalent to

$$P[\overline{X} - 1.96 \times \sigma/\sqrt{n} < \mu < \overline{X} + 1.96 \times \sigma/\sqrt{n}] = 0.95$$

That means that the interval

$$(\overline{X} - 1.96 \times \sigma/\sqrt{n}, \overline{X} + 1.96 \times \sigma/\sqrt{n})$$

includes the unknown parameter μ with a probability of .95.

Definition

The interval

$$(\overline{X} - 1.96 \times \sigma/\sqrt{n}, \overline{X} + 1.96 \times \sigma/\sqrt{n})$$

is **95% confidence interval for** μ *when the population is normal and* σ *known.*

Example 1. Calculating a CI for Normal Population σ Known

Example

Radiation of microwave ovens has normal distribution with standard deviation σ=0.6. A sample of 25 microwave ovens produced $\overline{X} = 0.11$. Determine a 95% confidence interval for the mean radiation.

Solution *The population is normal, and the observed value $\overline{X} = 0.11$.*

$$(0.11 - 1.96 \times 0.6/\sqrt{25}, 0.11 + 1.96 \times 0.6/\sqrt{25}) = (-.1252, 3452)$$

is a 95% confidence interval for μ.

$100(1-\alpha)\%$ confidence interval for μ

Definition

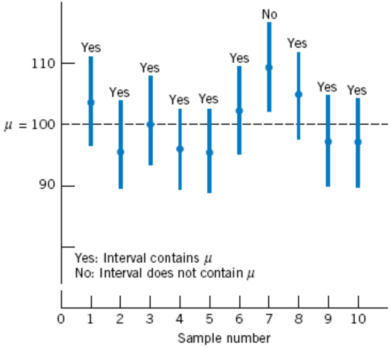
When the population is normal and σ *is known, a* $100(1-\alpha)\%$ **confidence interval for** μ *is given by*

$$(\overline{X} - z_{\alpha/2} \times \sigma/\sqrt{n}, \overline{X} + z_{\alpha/2} \times \sigma/\sqrt{n})$$

Interpretation of Confidence Intervals

Suppose we have

- Normal distributed population with $\mu = 100$ and $\sigma = 10$.
- Ten samples of size 7 are selected
- A 95% confidence interval $\overline{x} \pm 1.96 \times 10/\sqrt{7}$ is computed from each.



Interpretation of Confidence Intervals

Fact

1. *Before we sample, a confidence interval* $(\overline{X} - 1.96 \times \sigma/\sqrt{n}, \overline{X} + 1.96 \times \sigma/\sqrt{n})$ *is a random interval that attempts to cover the true value of the parameter* μ.
2. *The probability*

$$P[\overline{X} - 1.96 \times \sigma/\sqrt{n} < \mu < \overline{X} + 1.96 \times \sigma/\sqrt{n}] = 0.95$$

interpreted as the long-run relative frequency over many repetitions of sampling asserts that about 95% of the intervals will cover μ.

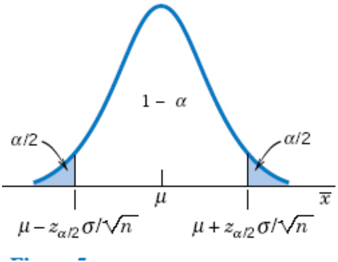
Interpretation of Confidence Intervals

Fact

1. *Once is calculated from an observed sample, the interval* $(\overline{x} - 1.96 \times \sigma/\sqrt{n}, \overline{x} + 1.96 \times \sigma/\sqrt{n})$ *which is a realization of the random interval, is presented as a 95% confidence interval for* μ*. A numerical interval having been determined, it is no longer sensible to speak about the probability of its covering a fixed quantity* μ*.*
2. *We* **never** *know if the 95% confidence interval covers the unknown mean* μ*.*

Large Sample Confidence Interval for μ.

Since the sample size is large ($n \geq 30$) we know that $\overline{X}$ has nearly normal distribution whatever the form of the population by **Central Limit Theorem**.



$$P[\overline{X} - z_{\alpha/2} \times \sigma/\sqrt{n} < \mu < \overline{X} + z_{\alpha/2} \times \sigma/\sqrt{n}] = 1 - \alpha$$

Large Sample Confidence Interval for μ.

If σ is unknown, we can replace $\sigma/\sqrt{n}$ with its estimator $S/\sqrt{n}$ because n is large. So,the large sample confidence interval for μ has the form

Estimate $\pm$ *(z Value)*$\times$ *(Estimated standard error)*

Definition

When n is large, a $100(1-\alpha)\%$ **confidence interval for μ** *is given by*

$$(\overline{X} - z_{\alpha/2} \times S/\sqrt{n}, \overline{X} + z_{\alpha/2} \times S/\sqrt{n})$$

where S is the sample standard deviation.

Example 2. Confidence Interval for mean time to process.

Example

A manager at a power company monitored the employee time required to process high-efficiency lamp bulb rebates. A random sample of 40 applications gave a sample mean time of 3.8 minutes and a standard deviation of 1.2 minutes. Construct a 90% confidence interval for the mean time to process μ.

Solution *For large n, a 90% confidence interval for μ is given by*

$$\overline{X} \pm z_{0.05} \times S/\sqrt{n}$$

Using $z_{0.05} = 1.645$, $n = 40$, $S = 1.2 minutes$, and $\overline{x} = 3.8 minutes$, the 90% confidence interval for μ (true mean processing time) is given by

$$3.8 \pm 1.645 \times 1.2/\sqrt{40} = 3.8 \pm 0.31 = (3.49, 4.11)$$

minutes.

Confidence Interval for a Parameter.

Definition

An interval (L, U) *is a* $100(1-\alpha)\%$ **confidence interval for a parameter** *if*

$$P[L < \mathit{Parameter} < U] = 1 - \alpha$$

and the endpoints L *and* U *are computable from the sample.*

Example 3. Confidence Interval for the amount of PCBs.

Example

The amount of PCBs (polychlorinated biphenyls) was measured in 40 samples of soil that were treated with contaminated sludge. The following summary statistics were obtained. $\overline{x} = 3.56$, $s = .5ppm$ *Obtain a 95% confidence interval for the population mean* μ, *amount of PCBs in the soil.*

Solution

For large n, a 95% confidence interval for μ *is given by*

$$\overline{X} \pm z_{0.025} \times S/\sqrt{n}$$

Using $z_{0.025} = 1.96$, $n = 40$, $S = .5ppm$, *and* $\overline{x} = 3.56ppm$, *the 95% confidence interval for* μ *(true mean amount of PCBs in the soil) is given by*

$$3.56 \pm 1.96 \times .5/\sqrt{40} = 3.56 \pm 0.155 = (3.405, 3.715).$$

For Next Time

- Read Section 8.3 from Johnson and Bhattacharyya
- Group Problems:

Group	1	2	3	4	5
Problem	8.15	8.17	8.19	8.21	8.25
Group	6	7	8	9	10
Problem	8.15	8.17	8.19	8.21	8.25