Math 630, Fall 2014
Homework #3: Due on 10/13 (A proper subset of the problems will be selected for grading.)
A) Solve the following problems from the textbook. 2.17, 2.33, 2.35, 2.36, 2.38, 2.45.
B) Suppose that \( f \) is a function on \( \mathbb{R} \times \mathbb{R}^k \) such that \( f(x, \cdot) \) is Borel measurable for each \( x \in \mathbb{R} \) and \( f(\cdot, y) \) is continuous for each \( y \in \mathbb{R}^k \). For \( n \in \mathbb{N} \), define \( f_n \) as follows. For \( i \in \mathbb{Z} \) let \( a_i = i/n \), and for \( a_i \leq x \leq a_{i+1} \) let

\[
f_n(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}.
\]

Prove that for each \( n \), \( f_n \) is Borel measurable on \( \mathbb{R} \times \mathbb{R}^k \), and that \( f_n \to f \) pointwise. Conclude that \( f \) is Borel measurable, and that every function on \( \mathbb{R}^n \) that is continuous in each variable separately is Borel measurable.
C) Prove that if \( f : \mathbb{R} \to \mathbb{R}^+ \) is such that \( f^{-1}((r, \infty)) \) is Lebesgue measurable for each \( r \in \mathbb{Q} \), then \( f \) is Lebesgue measurable.
D) Let \( C \subset [0, 1] \) be the Cantor set. Define \( f : [0, 1] \to [0, 1] \) such that each \( x = \sum_{j=1}^{\infty} a_j/3^j \in C \), \( (a_j \in \{0, 2\}) \) \( f(x) = \sum_{j=1}^{\infty} b_j/3^j \), where \( b_j = a_j/2 \).
D-1) Prove that \( f \) is increasing on \( C \) and construct an extension \( h \), of \( f \) to \([0, 1]\) such that \( h \) is increasing on \([0, 1]\). Prove that \( h \) is continuous on \([0, 1]\).
D-2) Define \( g(x) = x + h(x) \) for \( x \in [0, 1] \). Prove that \( g \) is a bijection from \([0, 1]\) onto \([0, 2]\), and let \( k = g^{-1} \). Prove that \( k \) is continuous from \([0, 2]\) to \([0, 1]\).
D-3) Prove that \( m(g(C)) = 1 \).
D-4) Let \( A \subset g(C) \) that is Lebesgue nonmeasurable, and set \( B = k(A) = g^{-1}(A) \). Prove that \( B \) is Lebesgue measurable but not Borel measurable.
D-5) Prove that there exist a Lebesgue measurable function \( F \) and a continuous function \( G \) on \( \mathbb{R} \) such that \( F \circ G \) is not Lebesgue measurable.
E) Let \( \{E_n\}_{n=1}^{\infty} \subset \mathcal{M}(\mathbb{R}) \), where \( (\mathbb{R}, \mathcal{M}) \) is the Lebesgue measurable space. Let \( E = \bigcup_{n=1}^{\infty} E_n \). Prove that \( f : E \to \mathbb{R} \) is measurable on \( E \) if and only if \( f|_{E_n} \) is measurable on \( E_n \) for each \( n \geq 1 \).
F) Let \( A, B \subset \mathbb{R} \). Prove that

\[
\text{F-1. } 1_{A \cap B} = 1_A1_B, \\
\text{F-2. } 1_{A \cup B} = 1_A + 1_B - 1_A1_B, \\
\text{F-3. } 1_{A^c} = 1 - 1_A, \text{ where } A^c \text{ is the complement of } A.
\]
G) Prove that \( \{f_n\}_{n=1}^{\infty} \) is a sequence of Lebesgue measurable functions from \( \mathbb{R} \to \mathbb{R} \), then the set \( S = \{x \in \mathbb{R}, \{f_n(x)\}_{n=1}^{\infty} \text{ converges} \} \) is measurable.
H) Let \( (\mathbb{R}, \mathcal{M}, m) \) denote the Lebesgue measure space on \( \mathbb{R} \). Let \( E \in \mathcal{M} \) with \( m(E) < \infty \).
H-1) Let \( f : E \to \mathbb{R} \) be a simple function. Prove that for every \( \epsilon > 0 \), there exists a compact set \( K \subset E \) with \( m(E \setminus K) < \epsilon \), and \( f \) is continuous on \( K \).
H-2) Prove that the result of part a) is still true if \( f \) is a Lebesgue measurable function on \( E \).
I) Solve the following problems from the supplement text posted on Canvas: 2.1.28, 2.1.30, 2.1.38, 2.1.39, 2.1.45. Solve the problems only for extended real-valued functions defined on \( \mathbb{R} \), i.e., assume \( d = 1 \). (Note that the exterior Lebesgue measure \(| \cdot |_e \) is the same as the Lebesgue outer measure \( m^* \) we defined in class).