Math 630, Fall 2014
Homework #4: Due on 10/27 (A proper subset of the problems will be selected for grading.)

A) Solve the following problems from the textbook. 3.1, 3.3, 3.8, 3.12, 3.15, 3.19, 3.26
(assume $X = [0, 1]$, $A = \mathcal{M}([0, 1])$, and $\mu = m$), 3.36.

B) Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space, and $f : \mathbb{R} \to [0, \infty]$ be a measurable function. Prove that $f = \sum_{n=1}^{\infty} a_n 1_{A_n}$, where $a_n \geq 0$, and $A_n \in \mathcal{M}$.

C) Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space, and $f : \mathbb{R} \to [-\infty, \infty]$ be a Lebesgue integrable function. Prove that for each $\epsilon > 0$, there exists $A \in \mathcal{M}$, $M(A) < \infty$ and
\[ \int_{\mathbb{R} \setminus A} |f| \, dm \leq \epsilon, \quad \text{and} \quad \sup_{x \in A} |f(x)| < \infty. \]

D) Consider the Lebesgue measure restricted to $[0, 1]$, and let $f : [0, 1] \to [-\infty, \infty]$ be a measurable function. For each $n \in \mathbb{N}$, define $A_n = \{ x \in [0, 1] : |f(x)| \geq n \}$. Prove that $f$ is Lebesgue integrable if and only if $\sum_{n=1}^{\infty} \mu(A_n)$ is convergent.

F) Consider the Lebesgue measure restricted to $[0, 1]$, and let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}([0, 1])$. For each $k \in \mathbb{N}$ let $B_k$ denote the set of all $x \in [0, 1]$ such that $x$ belongs to at least $k$ of the sets $A_n$.

F-1) Prove that $km(B_k) \leq \sum_{n=1}^{\infty} m(A_n)$.

F-2) Conclude that if the series $\sum_{n=1}^{\infty} m(A_n)$ converges, then $m-$ a.e. $x \in [0, 1]$ belongs only to finitely many $A_n$. 

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