Math 631, Spring 2013
Homework #4: Due on 3/12

1. Let \((X, \mathcal{A}, \mu)\) be a measure space, with \(\mu\) \(\sigma\)-finite. Let \(g\) be a \(\mu\)-measurable such that for all \(p \in [1, \infty)\) the mapping \(A\) given \(A(f) = fg\) maps \(L^p_\mu\) into \(L^1_\mu\).

   (a) If \(\int_X A(f) \, d\mu = 0\) for all \(f \in L^p_\mu\), prove that \(g = 0\) a.e.

   (b) Prove that if \(\{f_n\}_n\) is a sequence of \(\mu\)-measurable such that \(\lim_{n \to \infty} f_n = f, \mu\)-a.e., and \(|f_n| \leq h,\) \(h \in L^p\), then \(\lim_{n \to \infty} A(f_n) = A(f)\) in \(L^1_\mu\).

   (c) Prove that \(A\) is linear and bounded from \(L^p_\mu\) into \(L^1_\mu\), and conclude that \(g \in L^q_\mu\) where \(\frac{1}{q} + \frac{1}{p} = 1\).

2. Let \((X, \mathcal{A}, \mu)\) be a complete measure space, with \(\mu(X) = 1\), and such that for all \(A \in \mathcal{A}, \mu(A) \in \{0, 1\}\). Let \(\nu\) be a positive \(\sigma\)-finite measure on \((X, \mathcal{A})\) and assume that \(\nu \ll \mu\). Prove that \(\frac{d\nu}{d\mu} = \nu(X) \mu\)-a.e.

3. Let \((\mathbb{R}, \mathcal{M}, m)\) be the Lebesgue measure space, and \(1 < p, q < \infty\). Let \(T : L^{p_n}_{\mu_n} \to L^{q_n}_{\mu_n}\) be a bounded linear operator. Prove that there exists a unique bounded linear operator \(T' : L^{q_n'}_{\mu_n'} \to L^{p_n'}_{\mu_n'}\) such that

\[
\int_{\mathbb{R}} T(f)g \, dm = \int_{\mathbb{R}} fT'(g) \, dm
\]

for all \(f \in L^{p_n}_{\mu_n}, g \in L^{q_n}_{\mu_n}\), where \(\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1\).

4. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and \(1 \leq p \leq 2\). For \(f, g \in L^p_\mu(X)\), prove that

   (a) \(|f + g|^p + |f - g|^p \geq (|f|^p + |g|^p) + ||f||_p^p - ||g||_p^p|.

   (b) \((|f + g|^p + |f - g|^p)^p + ||f + g||_p - ||f - g||_p^p | \leq 2^p(||f||_p^p + ||g||_p^p).

Hint: [(b)] Follows from [(a)].
(For [(a)], let \(\alpha(r) = (1 + r)^{p-1} + (1 - r)^{p-1}\) for \(r \in [0, 1]\) and \(\beta(r) = [(1 + r)^{p-1} - (1 - r)^{p-1}]r^{1-p}\) for \(r \in [0, 1], \beta(0) = 0\) if \(p < 2\). Prove that for all \(A, B \geq 0, \alpha(r)|A|^p + \beta(r)|B|^p \leq |A + B|^p + |A - B|^p\). To establish this last claim, consider the function \(F_R(r) = \alpha(r) + B(r)R^p\), where \(R = \frac{|g|^p}{||g||_p^p} \leq 1\). What is the maximum of \(F_R\) on \(r \in [0, 1]\) and \(1 < p < 2\)?)