MATH 240 Homework Answer Key
Relieving the Pain # 1

Remarks: Please distinguish between the definitions of the terms and the theorems about the concepts behind these terms. Your goal as a student of Linear Algebra should be to master the ideas through examples in your exercises and through writing lucidly.

1. Put away the Turtle Wax. Here's some Tiger Balm.
a. any vector of the form $c_1v_1 + \cdots + c_kv_k$, where $c_1, \ldots, c_k$ are scalars.
b. the set of all linear combinations of the vectors $v_1, \ldots, v_k$.
c. the vector equation $x_1v_1 + \cdots + x_pv_p = 0$ implies $x_1 = \ldots = x_p = 0$.
d. $[\mathbf{a_1}|\cdots|\mathbf{a_n}]$; $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$; $x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n}$.
e. 1) $T(x + y) = T(x) + T(y)$ for all $x, y \in U$ and 2) $T(cu) = c \cdot T(u)$ for all $u \in U$ and all scalars c.
f. $T(x) = T(y) \Rightarrow x = y$.
g. for every $b \in V$ there is an element $x \in U$ such that $T(x) = b$. Or, you can say: $V = \text{range}(T)$

2. Some explanations, but no examples.
a. you can illustrate with an example involving two vectors. A geometrical picture is good, too.
b. You can draw or explain the different possibilities for the span of a collection of 2 vectors.
c. The vectors in a linearly independent set each give a 'fundamentally different' kind of direction. It's best to illustrate this with a picture. If you're so inclined, you can go through the logical argument why the definition we give is a good one for linear independence. For future reference, linear independence and the notion of span allow us to define precisely the *dimension* of a space.
d. You can take an appropriately sized matrix and vector and show how the product is a linear combination of the columns of the matrix.
e. Linear transformations are functions that preserve the operations of vector addition and scalar multiplication from one (vector) space to another. Thus, they are the functions that preserve linear combinations and 'respect the linear character' of the domain and codomain. It's a good idea to draw it as some of the exercises in section 1.8 and 1.9 ask.
f. Drawing a picture is easiest. One way to understand the concept of *one-to-one* mapping is that if you consider any element in the range (which is a subset of the codomain), there is one and only one element in the domain which maps to that element. Another way is to see that the range of a one-to-one mapping is a 'copy' of the domain that is a subset of the codomain.
Aside: A theorem tells us that a linear transformation T is one-to-one if and only if $T(x) = 0 \Rightarrow x = 0$. Also #4 shows a connection with *one-to-one* and

linear independence.
g. Again, a picture helps a lot. A mapping is onto if every element in the codomain has an element in the domain that maps to it. There might be more than one element in the domain that maps to a given element in the codomain. Another way is that the range of the mapping is the entire codomain.
Aside: #3 shows a connection with *onto* and a collection of vectors *span*ning $\mathbb{R}^m$.

3. Fill in the blanks with these!
a. m; n
b. Every; m; the columns of A.
c. has a solution $x \in \mathbb{R}^n$; every; m
d. m
e. n; m; onto.
not more (less than or equal to)

4. Fill in the blanks with these, too!
b. $x = 0$.
c. 1; 0
d. n
e. n; m; one-to-one.
$\geq$

Hum ho.
5. $\mathbf{x} + t\mathbf{y}$; $\quad \mathbf{v} + t(\mathbf{u} - \mathbf{v})$ or $t\mathbf{u} + (1-t)\mathbf{v}$

6. The solution set for $Ax = b$ is either empty or is a translation of the solution set for $Ax = 0$ by a vector u satisfying $Au = b$. So, in the case when $Ax = b$ is consistent, solutions take the form $x = u + n$, where n can be any vector satisfying $An = 0$ and u is a fixed vector satisfying $Au = b$.

7. They're not so tough after all, right?
a. $T\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{bmatrix} 13 \\ 4 \\ -10 \end{bmatrix}$; standard matrix of T is $\begin{bmatrix} 2 & -3 \\ 5 & 2 \\ 1 & 4 \end{bmatrix}$.
b. $T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, and the standard matrix of T is $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$;

$$T\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(-5) & + & 0 \cdot 3 \\ 2(-5) & + & (-1)3 \\ 0(-5) & + & 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -13 \\ 3 \end{bmatrix}.$$