Matlab Assignment 2
Math 246

due May 2
Read the instructions on the course web page.

1. We consider a pendulum with friction. Let \( y(t) \) denote the angle of a pendulum (measured from the bottom position) at time \( t \). Then \( y(t) \) satisfies the ODE \( y'' + 0.1y' + \sin(y) = 0 \). We start the pendulum at the bottom position \( y(0) = 0 \) and push it with an initial speed \( y'(0) = a \). Then the pendulum will either just swing back and forth before converging to the bottom position as \( t \to \infty \), or it may make a certain number \( k \) of full revolutions (going over the top) before converging to the bottom position. This means that \( y(t) \) converges to \( k \cdot 2\pi \) as \( t \to \infty \) where \( k \) is an integer which depends on the initial speed \( a \).

(a) For \( a = 2, 3, 4 \): Use ode45 and make a plot which shows the three curves \( y(t) \) vs. \( t \) for \( t \in [0, 50] \) together. What is the limit \( \lim_{t \to \infty} y(t) = k \cdot 2\pi \) in each case? Also make a phase plane plot which shows the three curves \( y_2 \) vs. \( y_1 \) together with the vector field (use vectfield).

(b) Determine values \( a > 0 \) so that we obtain \( \lim_{t \to \infty} y(t) = k \cdot 2\pi \) with \( k = 0, 1, 2, 3, 4 \). Make a plot which shows the five curves \( y(t) \) vs. \( t \) for \( t \in [0, 50] \) together. Also make a phase plane plot which shows the five curves \( y_2 \) vs. \( y_1 \) together with the vector field (use vectfield).

(c) Look at the phase plane plot from (b): What are the critical points? What is their stability?

2. Consider the initial value problem

\[
y'' + \gamma y' + 20y = f(t), \quad y(0) = 1, \quad y'(0) = 0
\]

with the following three forcing functions \( f(t) \)

\[
f_1(t) = 5 \sin(t), \quad f_2(t) = \begin{cases} 0 & \text{for } 0 < t < 6 \\ 10 & \text{for } 6 \leq t < 10 \\ 0 & \text{for } t \geq 10 \end{cases}, \quad f_3(t) = 40\delta(t - 6).
\]

(a) Let \( \gamma = 1 \). For each forcing function solve this problem in Matlab using the Laplace transform and the instructions on the web page. Print the solution and its Laplace transform in symbolic form. Plot the solution \( y(t) \) for \( t \in [0, 20] \). In each case explain the behavior of \( y(t) \): Why does \( y(t) \) correspond to the expected behavior of the mechanical system? What will happen for \( t \to \infty \)? What is the type of damping and how can you see this in the graphs?

(b) Now choose \( \gamma \) so that there is critical damping. Answer the questions from (a) and explain what is different.

(c) Let \( \gamma = 0.1 \) and use the forcing function \( f(t) = 5 \sin(\omega t) \). Choose \( \omega \) such that the steady state solution has maximal amplitude (see p. 209 in the textbook). Plot the solution. What happens for \( t \to \infty \)?

3. Consider the linear system \( \ddot{y} = Ay \) for the matrices

\[
(i) A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}, \quad (ii) A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \quad (iii) A = \begin{bmatrix} -0.5 & -1 \\ 2 & 1.5 \end{bmatrix}, \quad (iv) A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.
\]

(a) Use \([V,D]=\text{eig}(\text{sym}(\ldots))\) to find the eigenvalues and eigenvectors. State the type and stability of the critical point \((0,0)\).

(b) Use the code on the web page to draw a phase portrait: Use vectfield to plot the vector field for \( y_1 \in [-3, 3], y_2 \in [-3, 3] \), together with trajectories for various different initial values \( y_1^0 \in [-3, 3], y_2^0 \in [-3, 3] \) for \( t \) going from 0 to 30, and for \( t \) going from 0 to −30. Use \text{axis}([-3 3 -3 3]); \text{axis} equal after your plot commands.