MATH 241, WORKSHEET/QUIZ SOLUTIONS II

Quiz 3/16/06, Problem 1. Evaluate the integral $\iint_R y^2 dA$, where R is the region in the first quadrant bounded by the lines $y = 2x$ and $x + y = 2$, and the curve $y = x^2$.

Solution: There are two possible regions to choose. The solution here will be for the one at the origin, which is bounded below by $y = x^2$, and above by the lines $y = 2x$ and $y = 2 - x$. The lines intersect at $2x = 2 - x \implies x = \frac{2}{3}$; to the left of this point, the upper bound will be $2x$, while to the right of this point the upper bound is $2 - x$. Hence, the integral is:

$$\iint_R y^2 dA = \int_{x=0}^{\frac{2}{3}} \int_{y=x^2}^{2x} y^2 dy dx + \int_{x=\frac{2}{3}}^{1} \int_{y=x^2}^{2-x} y^2 dy dx.$$

Evaluating the integral is messy but straightforward.

Quiz 3/16/06, Problem 2. Find the volume of the solid region in the first octant that is common to the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. (*Hint*: Over what region in the xy plane does the region lie? Integrate first with respect to y.) What is the total volume of the region formed by this intersection?

Solution. We must figure out two things: what bounds the region on the sides, and what bounds the region on the top/bottom. The sides are given by the cylinder $x^2 + y^2 = 1$, and the top/bottom by the cylinder $x^2 + z^2 = 1$ or $z = \sqrt{1 - x^2}$. Since the region is in the first octant, the coordinate planes $x = 0$, $y = 0$, and $z = 0$ are also boundaries.

The limits of integration correspond to the disk $x^2 + y^2 \leq 1$ in the first quadrant, since this is the "shadow" of the solid region. The z function, which represents the height of the solid at a given (x, y), goes on the inside of the integral:

$$\begin{aligned} V &= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1 - x^2} dy dx \\ &= \int_0^1 \sqrt{1 - x^2} \cdot y|_0^{\sqrt{1-x^2}} dx = \int_0^1 (1 - x^2) \\ &= \left(x - \tfrac{1}{3}x^3\right)_0^1 = 1 - \tfrac{1}{3} - 0 = \frac{2}{3}. \end{aligned}$$

Worksheet 3/28/06, Problem 1.

Solution.

Worksheet 3/28/06, Problem 2.

Solution.

Worksheet 4/4/06, Problem 1. Set up the integral $\iiint_D xyz dV$, where D is the portion of the solid ball $x^2 + y^2 + z^2 \leq 4$ *outside* the cylinder $r = \cos\theta$. [Use cylindrical coordinates.]

Solution.

Worksheet 4/4/06, Problem 2. Set up the integral $\iiint_D e^{\sqrt{x^2+y^2}} dV$, where D is the solid region bounded above by the surface $z = 25 - x^2 - y^2$, below by the surface $z = x^2 + y^2 - 25$, and on the sides by the surface $x^2 + y^2 = 1 + \cos\theta$. [Use cylindrical coordinates.]

Solution.

Worksheet 4/4/06, Problem 3. Set up the integral $\iiint_D \frac{1}{x^2+y^2+z^2} dV$, where D is the solid region in the first octant bounded by the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$, and by the coordinate planes. [Use spherical coordinates.]

Solution.

Worksheet 4/4/06, Problem 4. Set up the integral $\iiint_D f(x, y, z) dV$, where f is twice the distance from (x, y, z) to the origin and D is the solid region bounded above by the cone $x^2 + y^2 = z^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the plane $z = -1$. [Use spherical coordinates.]

Solution.

Quiz 3/30/06, Problem 1. Find the volume of the solid region bounded above by the plane $z = 4 + x + 2y$, on the sides by the cylinder $x^2 + y^2 = 1$, and below by the xy plane.

Solution. This looks like a cylinder whose top is slanted. The "shadow" of the region is the unit disk, and polar coordinates are most natural here, so

$$\begin{aligned}\iint_R z\,dA &= \int_0^{2\pi}\int_0^1 (4 + r\cos\theta + 2r\sin\theta) r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 4r + r^2\cos\theta + 2r^2\sin\theta\,dr\,d\theta \\ &= \int_0^{2\pi} (2r^2 + \tfrac{1}{3}r^3\cos\theta + \tfrac{2}{3}r^3\sin\theta)|_0^1 d\theta = \int_0^{2\pi}(2 + \tfrac{1}{3}\cos\theta + \tfrac{2}{3}\sin\theta)d\theta \\ &= (2\theta + \tfrac{1}{3}\sin\theta - \tfrac{2}{3}\cos\theta)|_0^{2\pi} = 4\pi.\end{aligned}$$

Quiz 3/30/06, Problem 2. Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ above the plane $z = 5$.

Solution. The shadow of the surface is the disk where $z = 5 = 9 - x^2 - y^2 = 9 - r^2$, or $r^2 = 4 \implies r = 2$. Again, polar coordinates are the most natural. The integrand is

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}.$$

So the integral is (with u substitution $u = 1 + 4r^2 \implies du = 8r\,dr$):

$$\int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\,r\,dr\,d\theta = 2\pi\int_1^{17} \tfrac{1}{8}\sqrt{u}\,du = \frac{\pi}{4}(\tfrac{2}{3}u^{\frac{3}{2}})|_1^{17} = \frac{\pi}{6}(17^{\frac{3}{2}} - 1).$$

Quiz 4/6/06, Problem 1. Use a triple integral to find the volume of the region bounded below by the surface $z = \sqrt{r}$ and above by the plane $z = 1$.

Solution. To parametrize the integral, find the shadow of the solid in the xy-plane. This occurs at the intersection of the surfaces, hence where $\sqrt{r} = z = 1$, or $r = 1$ and $z = 1$. Hence, the integrals parametrize the unit disk. The third integral, corresponding to z, has the given surfaces as top and bottom.

The problem is most natural in cylindrical coordinates, so the integral is:

$$V = \iiint_D dV = \int_0^{2\pi} \int_0^1 \int_{\sqrt{r}}^1 r dz dr d\theta = 2\pi \int_0^1 (rz)|_{\sqrt{r}}^1 dr = 2\pi \int_0^1 r - r^{\frac{3}{2}} dr$$

$$= 2\pi(\tfrac{1}{2}r^2 - \tfrac{2}{5}r^{\frac{5}{2}})|_0^1 = 2\pi(\tfrac{1}{2} - \tfrac{2}{5}) = \frac{\pi}{5}.$$

Quiz 4/6/06, Problem 2. Find the total mass m of an object occupying the solid region bounded by the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 16$, with mass density at (x, y, z) equal to the reciprocal of the distance from (x, y, z) to the origin.

Solution. The integral is

$$\int_0^{2\pi} \int_0^{\pi} \int_2^4 \frac{1}{\rho^2} \rho^2 \sin\phi d\rho d\phi d\theta = 2\pi \int_0^{\pi} \int_2^4 \sin\phi d\rho d\phi$$

$$= 2\pi \int_0^{\pi} (4 - 2) \sin\phi d\phi = 4\pi(-\cos\phi)|_0^{\pi} = 8\pi.$$

Worksheet 4/11/06, Problem 1. Evaluate $\iint_R \left(\frac{x-4y}{x+4y}\right)^3 dA$, where R is the region bounded by the lines $x-4y=1$, $x-4y=5$, $x+4y=-1$, and $x+4y=2$.

Solution. Make the change of variables $u=x-4y, v=x+4y$. The Jacobian of this transformation is:

$$\frac{\partial(u,v)}{\partial(x,y)} = 17,$$

the determinant of the matrix of partial derivatives. Therefore, $dxdy = \frac{1}{17}dudv$. The region is bounded by $u=1,5$ and $v=-1,2$, hence the integral is:

$$\int_1^5 \int_{-1}^2 \left(\frac{u}{v}\right)^3 17dvdu,$$

which is easy to evaluate.

Worksheet 4/11/06, Problem 2. Let R be the region in the xy-plane bounded by the lines $x+3y=6$, $x-3y=6$, and $x=3$. Find the centroid of R.

Solution. There is no mass density, so the centroid is

$$(\bar{x},\bar{y}) = \left(\frac{\iint_R xdA}{\iint_R dA}, \frac{\iint_R ydA}{\iint_R dA}\right).$$

However, the region is symmetric, so we know immediately that $\bar{y}=0$. Compute:

$$\iint_R dA = \int_3^6 \int_{\frac{x}{3}-2}^{2-\frac{x}{3}} dydx = \int_3^6 4-\frac{2x}{3}dx = \left(4x-\frac{x^2}{3}\right)_3^6 = 24-12-12+3 = 3.$$

The integral weighted by x has the same limits:

$$\iint_R xdA = \int_3^6 \int_{\frac{x}{3}-2}^{2-\frac{x}{3}} xdydx = \int_3^6 4x-\frac{2x^2}{3}dx = \left(2x^2-\frac{2x^3}{9}\right)_3^6 = 72-48-18+6 = 12.$$

So the centroid is $(\frac{12}{3},0) = (4,0)$.

Be sure to include the density function $\delta(x,y,z)$ if it is given!