MATH 241, WORKSHEET/QUIZ SOLUTIONS

Worksheet 2/7/06, Problem 1. Find an equation of the plane tangent to the surface

$$\sin(xy) = 2 - z^2$$

at the point $(\pi, \frac{1}{2}, -1)$.

Solution: A tangent plane is determined by a point and a normal vector. The point has been given, $(\pi, \frac{1}{2}, -1)$. The normal vector is the normal to the surface at the given point.

There are two ways to compute this. First, if the surface is written

$$z = \pm\sqrt{2 - \sin(xy)} = f(x, y),$$

then the normal vector is $(-f_x, -f_y, 1)$. It is perhaps easier to recall that the gradient of a function $w(x, y, z)$ may be interpreted as a normal vector to the *level surface* $w(x, y, z) = C$. By moving all variables to one side of the equation for the surface, we can interpret it as the level surface of

$$w(x, y, z) = \sin(xy) + z^2 = 2.$$

The gradient of this function is

$$\nabla w = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k} = (y\cos(xy))\mathbf{i} + (x\cos(xy))\mathbf{j} + 2z\mathbf{k},$$

which evaluates at the given point to

$$(\tfrac{1}{2}\cos(\tfrac{\pi}{2}))\mathbf{i} + (\pi\cos(\tfrac{\pi}{2}))\mathbf{j} + 2(-1)\mathbf{k} = -2\mathbf{k},$$

the desired normal vector.

Therefore, the equation of the tangent plane is:

$$0(x - \pi) + 0(y - \tfrac{1}{2}) - 2(z + 1) = 0 \qquad \text{or simply} \qquad z = -1.$$

Worksheet 2/7/06, Problem 2. Find the extreme values of

$$f(x,y) = x^2 + y^2 - 2x - 4y - 6$$

in the disk specified by $x^2 + y^2 \leq 16$. [You need to use both Langrange Multipliers for the boundary and the First Derivative Test in the interior.]

Solution. Extreme values in a closed region occur either at critical points within the region, or at certain points on the boundary (determined by using the Lagrange Multipliers method).

The critical points occur when both $f_x = 0$ and $f_y = 0$. Since

$$f_x = 2x - 2, \qquad f_y = 2y - 4,$$

the only critical point is $(x, y) = (1, 2)$.

The boundary points to check are determined by the constraint

$$g(x,y) = x^2 + y^2 = 16$$

and the relation $\nabla f = \lambda \nabla g$ for some constant λ. In this case, the partials of g are $g_x = 2x$ and $g_y = 2y$, so the gradient relation becomes the two equations:

$$2x - 2 = \lambda(2x), \qquad\qquad 2y - 4 = \lambda(2y).$$

The next step is to use these two equations, together with the constraint, to solve for x and y. Isolate λ in each equation by dividing by $2x$ and $2y$:

$$\frac{2x-2}{2x} = \lambda = \frac{2y-4}{2y} \implies 1 - \frac{1}{x} = 1 - \frac{2}{y} \implies -\frac{1}{x} = -\frac{2}{y} \implies y = 2x.$$

Thus, the desired points satisfy both the equations $y = 2x$ and $x^2 + y^2 = 16$, meaning they occur at the intersection of a line ($y = 2x$) and a circle ($x^2 + y^2 = 16$). In particular:

$$x^2 + (2x)^2 = 16 \implies 5x^2 = 16 \implies x = \pm\sqrt{\tfrac{16}{5}} = \pm\tfrac{4}{\sqrt{5}}.$$

Since $y = 2x$, this means there are two points to check:

$$(x,y) = (+\tfrac{4}{\sqrt{5}}, +\tfrac{8}{\sqrt{5}}) \qquad \text{and} \qquad (x,y) = (-\tfrac{4}{\sqrt{5}}, -\tfrac{8}{\sqrt{5}}).$$

So the maximum/minimum values are determined by checking the values of f at these three points:

$$f(1,2) = -11, \qquad f(+\tfrac{4}{\sqrt{5}}, +\tfrac{8}{\sqrt{5}}) = 10 - \tfrac{40}{\sqrt{5}}, \qquad f(-\tfrac{4}{\sqrt{5}}, -\tfrac{8}{\sqrt{5}}) = 10 + \tfrac{40}{\sqrt{5}}.$$

Hence, the maximum is $10 + \frac{40}{\sqrt{5}}$ and the minimum is -11.

Quiz 3/2/06, Problem 1. Use the chain rule to compute the partial derivatives f_x and f_y where

$$f(u,v) = e^u \sin(v); \qquad u(x,y) = 2x + e^{3y}, \quad v(x,y) = x + e^{4y}.$$

Solution. The chain rule gives:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

The partials are:

$$f_u = e^u \sin(v), f_v = e^u \cos(v); \qquad u_x = 2, u_y = 3e^{3y}; \qquad v_x = 1, v_y = 4e^{4y}.$$

So f_x and f_y are given by:

$$f_x = 2e^u \sin(v) + e^u \cos(v) = 2e^{(2x+e^{3y})} \sin(x+e^{4y}) + e^{(2x+e^{3y})} \cos(x+e^{4y});$$

$$f_y = 3e^{3y}e^u \sin(v) + 4e^{4y}e^u \cos(v) = 3e^{3y}e^{(2x+e^{3y})} \sin(x+e^{4y}) + 4e^{4y}e^{(2x+e^{3y})} \cos(x+e^{4y}).$$

Quiz 3/2/06, Problem 2. A beetle is crawling on a sphere upon which the temperature is given by the function

$$T(x,y,z) = 2xy^2 - z^3x.$$

If its path is given by the curve $(x(t), y(t), z(t)) = (\frac{3}{5}\cos t, \frac{3}{5}\sin t, \frac{4}{5})$, find the rate of change of temperature, from the ant's point of view, when $t = \pi/2$.

Solution. Okay, so there's both an ant and a beetle in this problem. Maybe we should just call it a bantle.

The key to this is figuring out what is being asked. The *rate* of change in temperature is $\frac{dT}{dt}$, the derivative of *Temperature* with respect to *time*, and is computed using the chain rule:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = (2y^2 - z^3)(-\tfrac{3}{5}\sin t) + 4xy(\tfrac{3}{5}\cos t) - 3z^2x(0).$$

At the given time $t = \frac{\pi}{2}$, $(x,y,z) = (0, \frac{3}{5}, \frac{4}{5})$, so:

$$\frac{dT}{dt}(\frac{\pi}{2}) = (2(\tfrac{3}{5})^2 - (\tfrac{4}{5})^3)(-\tfrac{3}{5}) + 4(0)(\tfrac{3}{5})(0) = (\tfrac{18}{25} - \tfrac{64}{125})(-\tfrac{3}{5}) = \frac{-78}{625}.$$

Worksheet 2/28/06, Problem 9. Label surfaces shown and give a possible equation for each ($x^2 + y^2 + z^2 = 1$, $x^2 + y^2 = z^2$, etc.)

Solution. There are many possible equations for each, depending on the constants given. When all constants are 1, the solutions are as below:

(a) *Cylinder* about the z-axis, so $x^2 + y^2 = 1$;

(b) *Ellipsoid*, so $x^2 + y^2 + z^2 = 1$;

(c) *Cone* about the x-axis, so $x^2 = y^2 + z^2$;

(d) *araboloid* about the *negative* y-axis, so $y = -(x^2 + y^2)$;

(e) *Hyperboloid of 1 Sheet* about the z-axis, so $z^2 = x^2 + y^2 + 1$;

(f) *Two Hyperbolic Sheets* about z-axis, so $y^2 - x^2 = 1$ (note that the curves intercept the y-axis and not the x-axis);

(g) *Hyperbolic Paraboloid* with respect to z-axis, so $z = x^2 - y^2$ (note that positive z-values have intercepts with the x-axis);

(h) *Cone* about the line $y = x$. The equation for this is difficult to guess (harder than you should ever see). One possibility is $(x - y)^2 = z^2 + (x + y)^2 \implies z^2 = -4xy$.