Complex Analysis Summary (Level II)

Elisha Peterson

December 19, 2002

Contents

1 Introduction

1.1 The Origin of Complex Numbers

dhh 1.1-2.2

The natural numbers $\mathbb{N}$ can be defined using sets. The integers $\mathbb{Z}$ can be defined as solutions to such equations as $x + 5 = 2$, rationals $\mathbb{Q}$ as solutions to $ax = b$, the reals $\mathbb{R}$ as solutions to $x^2 = 2$ (although the notion of distance in analysis leads us to define them as sequences of rational numbers), and finally the complex numbers $\mathbb{C}$ as solutions to $x^2 = -1$.

The concept of integrability is based upon limits of Riemann sums over real functions, and trigonometric functions can be defined as integrals, for the sake of rigor, and proven to satisfy the known identities.

The complex numbers were defined in order to solve arbitrary polynomial equations. They can be written $z = x + iy = re^{i\theta}$. θ is called the *argument*, and is unique up to multiples of 2π. If we restrict θ to $(-\pi, \pi]$, we obtain the *Argument*.

1.2 The Riemann Sphere

dhh 2.3

By projecting the standard sphere $S^2 \subset \mathbb{R}^3$ onto the plane (stereographic projection), we obtain another model of $\mathbb{C}$ as the sphere minus one point, called the point at infinity. This projection is conformal, maps circles to circles/lines, and is given by $P : S^2 \to \mathbb{C}$:

$$\begin{aligned} P(x_1, x_2, x_3) &= \frac{x_1 + ix_2}{1 - x_3}; \\ P^{-1}(x + iy) &= \frac{1}{1 + |z|^2}(2x, 2y, |z|^2 - 1). \end{aligned}$$

Stereographic projection induces metrics on the plane given by the standard chordal or spherical metrics on the sphere.

Defining complex functions on the sphere allows them to take the value ∞; with this viewpoint in mind, the sphere is the *extended complex plane* $\hat{\mathbb{C}}$.

1.3 Bilinear Transformations

dhh 2.4

Bilinear or *Möbius Transformations* are a widely used class of functions on $\mathbb{C}$ of the form:

$$F(z) = \frac{az + b}{cz + d}.$$

Such maps are *conformal*, meaning they preserve angles between curves, and actually every conformal map on $\hat{\mathbb{C}}$ (the Riemann sphere) has this form. They are composed of rotations, translations, and inversions, and preserve the class of circles and lines. They are uniquely defined by their action on three points.

A useful tool for computing specific Möbius transformations is the *crossratio*:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The crossratio is invariant under bilinear transformations. It is real iff the four given points lie on a circle or a line. One special case is the map $S(z) = (z, z_2, z_3, z_4)$, which takes z_2 to 1, z_3 to 0, and z_4 to ∞, so can be used to find the unique bilinear transformation taking any three points to any other three points.

If one looks at the sphere algebraically as the complex projective plane $\mathbb{C}P^2$, defined as $\mathbb{C}^2 \backslash_{\{(0,0)\}}$ with equivalence $(z, w) \sim (a, b)$ if $z/w = a/b$, then bilinear transformations are just linear maps (matrices) acting on $\mathbb{C}^2$. Moreover, the composition of maps is given by matrix multiplication.

2 Complex Functions

2.1 Convergence in the Complex Plane

dhh 3.0, 3.6, 5.9

One of the most common notions of convergence in the complex plane is *uniform convergence (on a disk)*, i.e., $\lim_{z\to\zeta} f(z) = w$ uniformly if $\forall\epsilon > 0, \exists\delta > 0$ such that $|z - \zeta| < \delta$ implies $|f(z) - w| < \epsilon$. Other notions of convergence include pointwise convergence, uniform convergence on the whole domain, and absolute convergence.

Normal convergence is often used when working with generalized series of functions. A series converges *normally* in a region R if it converges uniformly on every relatively compact subset of the region. Just as the uniform limit of continuous functions is continuous, so if a sequence of analytic functions $f_n(z) \to f(z)$ normally, then $f(z)$ is analytic, and $f_n'(z) \to f'(z)$ normally.

A useful test for normal convergence is the *Weierstrass M-test*: a series of functions $\sum f_n(z)$ analytic on a region converges normally provided for every closed ball B in the region, there exists M_n such that $|f_n(z)| \le M_n\ \forall z \in B$ and $\sum M_n < \infty$.

2.2 General Classes of Functions

Holomorphic Functions

dhh 3.0-3.3

There are many ways to differentiate a given complex function $f(z) = u(x,y) + iv(x,y)$:

- *Real partials*: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$. These combine to form the *Jacobian* Df.
- *Complex-valued partials*: $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, which combine to give the *differential* $df = f_x dx + f_y dy$.
- *Complex partials*: $\partial f = f_z = \frac{1}{2}(f_x - if_y)$ and $\bar\partial f = f_{\bar z} = \frac{1}{2}(f_x + if_y)$. For the most part, these operators act on functions of z and $\bar z$ as they would on any real function of x and y.
- f is *differentiable* at z if there exist $a, b \in \mathbb{C}$ such that

$$\lim_{\zeta\to 0} \tfrac{f(z+\zeta) - f(z) - a\zeta - b\bar\zeta}{\zeta} = 0.$$

- If $b = 0$, then the limit below converges and f has *complex derivative* $f'(z)$:

$$f'(z) = \lim_{\zeta\to 0} \tfrac{f(z+\zeta) - f(z)}{\zeta}.$$

This last condition is the most useful. It is equivalent to both $\bar\partial f = 0$ and the *Cauchy-Riemann equations*: $u_x = v_y$ and $u_y = -v_x$. Such functions are called *holomorphic* and can be differentiated with respect to z just like any function of a real variable.

Conformal Functions

dhh 4.10

A differentiable map f is *conformal* at a point z if it preserves angles, i.e., if whenever two curves α and β intersect at z with angle θ, $f(\alpha)$ and $f(\beta)$ intersect at $f(z)$ with angle θ. This condition is equivalent to the Cauchy-Riemann equations, so a map is conformal iff it is holomorphic with nonzero derivative (or locally just holomorphic and $1:1$).

Any two simply-connected proper subsets of $\mathbb{C}$ are conformally equivalent (the *Riemann Mapping Theorem*). There is a standard dictionary of conformal maps taking disks to half-planes, strips, half-strips, and more.

dhh 3.6-3.9

Analytic Functions

A power series is just an infinite series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. There is a *radius of convergence* R ($0 \le R \le \infty$), such that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$. *Abel's First Theorem* states that

$$R = \frac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}.$$

Nothing can be said for the points $|z| = R$, although $f(z)$ cannot converge everywhere on this circle.

One can differentiate a power series termwise to obtain $f'(z) = \sum_{k=0}^{\infty} k a_k z^{k-1}$, which has the same radius of convergence. This shows that coefficients of power series are unique and given by $a_k = f^k(0)/k!$. Other operations on power series that respect convergence include addition, multiplication, division, composition, and antidifferentiation.

A function f is *analytic* at a point ζ if $\exists R > 0$ such that $\sum_{k=0}^{\infty} a_k(z-\zeta)^k$ converges to $f(z)$ for $|z - \zeta| < R$. Analytic and holomorphic are equivalent conditions (the proof that holomorphic $\implies$ analytic will come later).

Abel's Second Theorem says that any power series $\sum a_k z^k$ converging at ζ is an analytic function $f(z)$ for $|z| < |\zeta|$ and

$$\lim_{r\to 1} f(r\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k.$$

2.3 Special Classes of Functions

dhh 3.4

Polynomial Functions

Complex polynomials of degree n have the form $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, with $a_i \in \mathbb{C}$. There is a division algorithm, based on degrees, for complex polynomials. The *Fundamental Theorem of Algebra* asserts the existence of a complex root of any polynomial. (By factoring a polynomial, the existence of one root implies the existence of n roots, counting multiplicity.) *Lucas' Theorem* for polynomials states that roots of $p'(z)$ lie in the convex hull of the roots of $p(z)$, a sort of generalization of Rolle's Theorem for real functions.

dhh 3.5

Rational Functions

If $P(z), Q(z)$ are polynomials with degrees p, q, then $R(z) = P(z)/Q(z)$ is a *rational function* with degree $n = \max\{p, q\}$, and covers every point n times. $R(z)$ is *meromorphic*, meaning it acts holomorphically on the extended complex plane $\hat{\mathbb{C}}$.

Rational functions may be factored into partial fractions to give

$$R(z) = \frac{P(z)}{Q(z)} = P_0(z) + \sum_{k=1}^{m} \frac{P_k(z)}{z - \beta_k}.$$

If $P_k(z) = c_k$ is constant, this coefficient can be computed using L'Hôpital's Rule:

$$c_k = \lim_{z\to\beta_k} (z - \beta_k) R(z) = \frac{P(\beta_k)}{Q'(\beta_k)}.$$

dhh 3.10

Trigonometric and Exponential Functions

The complex exponential, sine, and cosine functions are defined by their power series:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k;$$
$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k};$$
$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

Expected identities are easily verified using these definitions.

dhh 3.11

Inverse Functions

We can prove an *Inverse Function Theorem* for $\mathbb{C}$: if $f(z)$ is analytic at a and $f'(a) \neq 0$, then f is $1:1$ on some disk $|z-a| < r$ and has inverse $g(w)$ analytic on some disk $|w - f(a)| < s$ (so $f \circ g(w) = w$ on this disk).

This allows us to invert $\exp(z)$ on any region where it is $1:1$ to obtain the complex logarithm. Since 0 is the only critical point of $\exp(z)$, we can define $\log(z)$ on any region of $\mathbb{C}$ which leaves out an arc extending from the origin. The choice of such an arc is called a *branch* of the logarithm, with *principal branch* given by $\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z)$ on $\mathbb{C}\backslash_{(-\infty,0]}$. It has power series

$$\mathrm{Log}(z) = \mathrm{Log}(\zeta) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\zeta^k}(z-\zeta)^k,$$

converging for $|z-\zeta| < |\zeta|$.

Many other functions can be defined using the complex log (and so thus their definition depends on the choice of branch). For example, the arctangent can be defined on $\mathbb{C}\backslash_{\{iy:|y|\geq 1\}}$ by

$$\mathrm{Arctan}(z) = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right).$$

Complex roots and powers are defined on the principal branch of log, with $z^{1/n} = r^{1/n}\left(\cos(\frac{\theta}{n}) + i\sin(\frac{\theta}{n})\right)$, or more generally $z^a = \exp(a\mathrm{Log}(z))$ for any $a \in \mathbb{C}$. A consequence of these definitions is Euler's formula $e^{i\pi} + 1 = 0$.

3 Topological Properties

3.1 The Maximum Principle

dhh 4.2-4.4

A fundamental theorem of complex analysis is the *Maximum Principle*, which states that an analytic function $f(z)$ defined on some domain has maximum modulus $|f(z)|$ on the boundary of that domain.

The Maximum Principle follows from the *Open Mapping Theorem* (nonconstant analytic maps are open), since if z is an interior point, then $f(z)$ must also be interior, so $|f(z)|$ cannot be maximum. The Open Mapping Theorem follows in turn from the Inverse Function Theorem (which implies maps are open for all z but where $f'(z) = 0$), and the fact that maps such as z^n are open (which takes care of the points when $f'(z) = 0$).

An immediate consequence is the Fundamental Theorem of Algebra: if a polynomial $p(z)$ had no zeros, then $1/p(z)$ would be analytic on $\mathbb{C}$, and since $\lim_{|z|\to\infty} |1/p(z)| = 0$, the Maximum Principle would imply $1/p(z) = 0$ for all z, an absurdity.

3.2 The Schwarz Lemma

dhh 4.5,4.7

Another consequence of the maximum principle: if f is analytic in the unit disk with $|f(z)| \leq 1$ and $f(0) = 0$, then either $|f(z)| < |z|$ for $z \neq 0$ or $f(z) = e^{i\varphi} z$.

More generally, if f is bounded by M on $\{|z - w| < R\}$, then

$$\left| \frac{M(f(z) - f(w))}{M^2 - f(z)\overline{f(w)}} \right| \leq \left| \frac{R(z - w)}{R^2 - z\bar{w}} \right|.$$

Equality holds only when f is bilinear. As $z \to w$ we have:

$$\rho(\zeta) = \frac{2|d\zeta|}{1 - |\zeta|^2} \leq \frac{2|dz|}{1 - |z|^2} = \rho(z),$$

where $\rho(z)$ is, by definition, the *hyperbolic (Poincaré) metric*. Thus, for a curve γ we have $\rho(f \circ \gamma) \leq \rho(\gamma)$.

3.3 Analytic Continuation

dhh 4.8

Analytic functions have *isolated zeros*, meaning two analytic functions which agree on a convergent set of points must be equal in a neighborhood of the limit point. Also, an analytic function f with $f^k(a) = 0, \forall k$ is identically zero on a neighborhood of a (equivalently, power series expansions about a given point are unique).

One consequence of this uniqueness is that almost all trig identities true in $\mathbb{R}$ must also be true in $\mathbb{C}$. Another consequence is *analytic continuation*: if two analytic functions f and g are defined on domains D_f and D_g, respectively, with $f = g$ on $D_f \cap D_g \neq \emptyset$, then we can extend f to $D_f \cup D_g$ by letting $f(z) = g(z)$ for $z \in D_f$. Thus, an analytic function is determined by its local behavior.

4 Complex Integration

dhh 5.1-5.2

Some Basic Facts on Complex Integration

The *line integral* of a complex function f over a curve $\lambda : [0,1] \to \mathbb{C}$ is given by:

$$\int_\lambda f dz = \int_0^1 f(\lambda(t))\lambda'(t)dt.$$

An *integral transform* takes a function f integrable on a measure space X to functions analytic on a region $R \subset \mathbb{C}$ by integrating:

$$g(z) = \int_X K(z,x)f(x)dx.$$

If the *kernel* $K(z,x)$ is analytic in z in R, with power series coefficients integrable in x, then the transform $g(z)$ is also analytic in R.

4.1 Cauchy's Integral Formula

dhh 5.4-5.7

There is a version of *Green's Theorem* for complex functions: if $\bar{\partial} f$ is continuous on a region R, then

$$\int_{\partial P} f dz = 2i \int\int_P \bar{\partial} f dxdy,$$

where P is a polygon contained in R. Of course, this generalizes to smooth curves. Note that if λ bounds a region on which f is holomorphic, then $\bar{\partial} f = 0$, so that $\int_\lambda f(z)dz = 0$. This special case is sometimes called *Cauchy's Theorem*.

A generalization of this result is *Cauchy's Integral Formula*, which states that for f integrable on a disk D with λ a positively oriented circle in D,

$$\int_\lambda \frac{f(\zeta)}{\zeta - z_0} d\zeta = \begin{cases} 2\pi i f(z_0), & z_0 \text{ inside } \zeta; \\ 0, & z_0 \text{ outside } \zeta. \end{cases}$$

This shows that holomorphic functions are actually analytic, since this is really an integral transform with analytic kernel $1/(\zeta - z_0)$. Differentiating, we see that f has a *Taylor Series* expansion absolutely convergent on compact subsets of D:

$$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

4.2 The Mean-Value Theorem

dhh 5.8

Cauchy's formula can be used to limit f and its derivatives. By evaluating around a circle about z_0, we have the *Mean-Value Theorem*:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Differentiating the original Cauchy formula gives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^n e^{in\theta}} d\zeta,$$

so that $|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_\theta |f(z_0 + re^{i\theta})|$.

In particular, if f is bounded by M on $\{|z - z_0| = r\}$, then $|f'(z_0)| \leq \frac{M}{r}$. Letting $r \to \infty$, we see that any bounded/entire function must be constant (*Liouville's Theorem*).

4.3 A Generalized Cauchy Formula

dhh 7.1-7.5

Given a closed curve (or cycle) α, the *index* or *winding number* of z_0 with respect to α is

$$n(z_0, \alpha) = \frac{1}{2\pi i} \int_\alpha \frac{d\zeta}{\zeta - z_0}.$$

This is the measure of variation in argument (or angle) around a curve; intuitively, it measures how many times the curve winds around z. It is integer-valued, and constant on each component of $\mathbb{C} \setminus \alpha$.

This gives a way to define homology classes in a region R of the complex plane. Namely, a cycle α in R is *homologous to* 0 if $n(z, \alpha) = 0$ for $z \notin R$; two cycles α, β are *homologous* if $\alpha - \beta$ is homologous to 0.

The integral of a function over a cycle α depends only on the homology class of α, allowing us to generalize Cauchy's Formula. Given α homologous to 0 in a region R, and a function f analytic on R, we have *Cauchy's Thorem* $\int_\alpha f(z)dz = 0$, and *Cauchy's Formula*

$$\int_\alpha \tfrac{f(\zeta)}{\zeta - z_0} d\zeta = 2\pi i f(z_0) n(z_0, \alpha).$$

5 Residue Theory

5.1 The Laurent expansion

dhh 7.6

One application of Cauchy's Formula is the *Laurent expansion*: a function $f(z)$ analytic on the annulus $A = \{r < |z - z_0| < R\}$ can be expanded in a power series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where both the positive and negative series are normally convergent on A, and

$$a_k = \frac{1}{2\pi i} \int_\lambda \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta,$$

where λ is chosen so that $n(z_0, \lambda) = 1$.

The proof uses Cauchy's Formula to express $f(z) = \int_\alpha \diamondsuit - \int_\beta \heartsuit$, where α is outside z, β inside z, with $\alpha - \beta$ null-homologous and $n(z, \alpha - \beta) = 1$. $\int_\alpha \diamondsuit$ is then expanded into a series with negative powers, while $\int_\beta \heartsuit$ is expanded into one with positive powers. Summing the two gives the Laurent expansion.

The 'non-analytic' half $\sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k}$ of the Laurent expansion is called the *principal part* of the expansion, and the coefficient a_{-1} is called the *residue (at z_0)*.

Singularities of Complex Functions

dhh 7.7-7.8

A singularity z_0 of f is an *isolated singularity* if f is analytic on some 'punctured neighborhood' $U \backslash_{\{z_0\}}$. *Riemann's Lemma* implies that f is unbounded on this set (otherwise f would be analytic at z_0).

There are several types of isolated singularities. If the *residue* a_{-1} is the only nonzero term in the principal part of the expansion, z_0 is called a *simple pole*. More generally, if $a_{-k} = 0$ for $k > N$ and $a_{-N} \neq 0$, then z_0 is a *removable singularity* or a *pole of order N*. Otherwise, z_0 is a *transcendental singularity*.

5.2 The Residue Theorem

The *Residue Theorem* states that for a cycle α bounding a region D with f analytic on the closure of D except for isolated singularities $b_1, \ldots, b_k$ in D, the integral of f over α is:

$$\int_\alpha f(z) dz = 2\pi i \sum_{i=1}^{k} \mathrm{res}(f, b_i).$$

Thus, only the residues of the function at each b_i need to be computed. If b_i is a simple pole, we can write $f(z) = \frac{p(z)}{q(z)}$ where $q(b_i) = 0$ and $p(b_i) \neq 0$, allowing the quick calculation:

$$a_{-1} = \lim_{z \to b_i} (z - b_i) f(z) = \frac{p(b_i)}{q'(b_i)}.$$

5.3 Applications of Residues

Applications to Complex Analysis

Applying the Residue Theorem to $\frac{f'(z)}{f(z)}$, we obtain the *Argument Principle*: if α is a cycle bounding a region D, and $f(z)$ is meromorphic on the closure of D, then $n(0, f(\alpha)) = N - M$, where f has N zeros and M poles in D. This implies that an analytic function f with $f(\alpha)$ bounding a region $\hat{D}$ must be $1:1$ on D, since for $\hat{w} \in \hat{D}$, the function $f(z) - \hat{w}$ has a unique zero.

The Argument Principle also implies *Rouchés Theorem*: given f, g analytic in and on a bounding cycle α, with $|f| > |g|$ on α, then f and $f + g$ must have the same number of zeros inside α. Drawing a picture, one sees intuitively that adding g to f cannot change its winding number about 0. Alternately, the Argument Principle implies that $n(0, 1+g/f) = 0$, so the number of poles (zeros of f) equals the number of zeros (zeros of $f + g$).

Applications to Calculus

Many definite integrals are impossible to solve using real analysis, but can be easily computed using the Residue Theorem. Generally, the method of solution is to make a change of variables (for an integral involving $\cos(\theta)$, one can substitute $z = e^{i\theta}$ so that $\cos(\theta)$ becomes $\frac{1}{2}(z + \frac{1}{z})$), and evaluate the new integral in the complex plane using residues. Improper integrals are also well-suited to complex analysis, as are certain infinite sums (by choosing a function whose residues represent the sum). These applications are discussed further in **??**.

6 Harmonic Functions

6.1 The Harmonic Mean-Value Property

A real-valued function $u(z)$ is *harmonic* if $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$, or equivalently, for $z = x + iy$,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The Cauchy-Riemann equations imply that the real and complex parts of an analytic function are harmonic. And, if u is harmonic on a disk D, then there exists an analytic function f on D with $u = \Re(f)$.

Taking the real part of the mean value theorem for analytic functions gives the *harmonic mean-value property*:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

It follows that harmonic functions satisfy a *maximum principle*: if u is harmonic on a region R with $\limsup_{z \to \partial R} u(z) \leq M$, then $u \leq M$ everywhere on R.

6.2 The Dirichlet Problem

An important problem for harmonic functions is the *boundary-value (Dirichlet) problem*: given a region R and a function h on ∂R, find a harmonic function u on R with boundary value h.

Using the mean-value property and a Möbius transformation, one can solve the Dirichlet problem on the disk given a real-valued, integrable function $h(e^{i\theta})$ on the boundary of the disk:

$$u(z) = \int_0^{2\pi} p(z, \phi) h(e^{i\phi}) d\phi,$$

where $p(z, \phi)$ is the *Poisson kernel*:

$$p(z, \phi) = p(re^{i\theta}, \phi) = \tfrac{1}{2\pi} \left(\tfrac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \right) = \tfrac{1}{2\pi} \Re \left(\tfrac{1+ze^{-i\phi}}{1-ze^{-i\phi}} \right).$$

Being able to solve the Dirichlet problem on the disk implies a solution for any region R conformally equivalent to a disk. In particular, by the Riemann mapping theorem, this means that it can be solved for any non-trivial simply-connected region, i.e., for any region bounded by a Jordan curve.

6.3 Schwarz Reflection

Because the Poisson integral $u(z) = \int p(z, \phi) h(e^{i\phi}) d\phi$ is harmonic, we have that any continuous function satisfying the mean-value property on a region R is harmonic.

As an application, a harmonic function v defined on $R \subset \{\Im(z) \geq 0\}$ intersecting the real line at an interval I with $v|_I = 0$, can be extended to a harmonic function on the symmetric domain $R \cup \bar{R} = R \cup \{z : \bar{z} \in R\}$. By defining $v(z) = -v(\bar{z})$ on $\bar{R}$. One can show that the mean-value property is satisfied everywhere on $R \cup \bar{R}$, hence v is harmonic.

This can be extended to give the *Schwarz Reflection Principle* for analytic functions: given an analytic function f on R which is real-valued on I, one can extend f to an analytic function on $R \cup \bar{R}$ with $f(\bar{z}) = \overline{f(z)}$. The reflection may also be in any arc of a circle or line segment.

7 Normal Families

7.1 Normal Convergence

Recall our definition of *normal convergence*: $f_n \to f$ normally on a domain $D \subset \mathbb{C}$ if $f_n \to f$ uniformly on every compact subset of D. Many properties of the functions f_n are preserved under normal convergence:

- If the f_n are analytic, f is also analytic;
- The zeros of f are the limits of zeros of the f_n, in the sense that if $f_n \to f$ normally, and ζ is a zero of f of multiplicity m, then $\forall r > 0$ there exists N such that for $n > N$, f_n has exactly m zeros in the disk $\{|z - \zeta| < r\}$;
- If the f_n are conformal, f is constant or conformal (combining the two previous results).

7.2 General Families of Functions

Let $\mathcal{F}$ be a family of functions on a compact set E. The following are standard definitions:

Equicontinuous for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$, for all $f \in F$ (essentially, every function in the family has the same degree of continuity);

Bounded there exists $M > 0$ such that $\sup_{f \in \mathcal{F}, x} |f(x)| < M$.

The *Arzela-Ascoli Theorem* says that $\mathcal{F}$ is relatively sequentially compact in the uniform norm (that is, every sequence of functions has a uniformly convergent subsequence) iff $\mathcal{F}$ is both equicontinuous and bounded.

7.3 Normal Families

Denote the set of functions analytic on a domain D by $H(D)$, and let $\mathcal{F} \subset H(D)$ be a family of functions contained therein. Some possible properties of $\mathcal{F}$:

Normal for every sequence $f_n \in \mathcal{F}$ there is a normally convergent subsequence;

Closed if $f_n \to f$ with $f_n \in \mathcal{F}$, then $f \in \mathcal{F}$;

Compact both normal and closed;

(Locally) bounded for all compact $E \subset D$ there exists M such that $sup_E|f(z)| < M$ for all $f \in \mathcal{F}$.

In the case of analytic functions, *Montel's Theorem* says that $\mathcal{F}$ is normal iff it is (locally) bounded, so that equicontinuity is unnecessary. Hence, $\mathcal{F}$ is compact iff it is (locally) bounded and closed.

For meromorphic functions (analytic except for isolated singularities), there is a similar result: $\mathcal{F} \subset M(D)$ is normal with respect to the spherical metric iff for every compact $E \subset D$, $\sup_E \frac{|f'(z)|}{1+|f(z)|^2} < K(E) < \infty$ uniformly over $\mathcal{F}$.

7.4 Normal Regions

One can also ask for what regions a given family $\mathcal{F}$ is normal, hence where are we guaranteed convergence.

Given a rational function f, the *Fatou set* $\mathcal{F}$ is the largest open set in which the iterates f^n form a normal family. Its (closed) complement $\mathcal{J}$ is the *Julia set.* Any rational f with $\deg(f) > 1$ has nonempty Julia set, and for polynomials $\mathcal{J}$ is compact.

8 Conformal Mappings

Conformal mappings are the $1 : 1$ analytic functions on some domain, and preserve the angles between curves.

8.1 The Riemann Mapping Theorem

This is one of the most famous and important results in complex analysis: given any simply connected domain $D \neq \mathbb{C}$ and $\zeta \in D$, there is a unique conformal map f from the unit disk $\mathbf{D}$ onto R such that $f(0) = \zeta$ and $f'(0) > 0$.

9 Unplaced Sections (Topics?)

9.1 Riemann Surfaces

dhh 4.9

A function, such as $\sqrt{z}$, not well-defined on $\mathbb{C}$, can be mapped into a surface (a complex manifold called the *Riemann surface*) on which it is well-defined and analytic.

How is this surface constructed? In general, a nonconstant analytic function f with domain D is locally $1:1$ away from its critical points $f'(z) = 0$. A Riemann surface for f is constructed by partitioning the domain into a countable number of regions $\{D_k\}$ on which f is $1:1$. Then, we glue the regions $R_k = f(D_k)$ along the boundary curves $f(\partial D_k)$. For example, z^2 is $1:1$ on $\{\Im(z) > 0\}$ and $\{\Im(z) < 0\}$. These regions both map to $U = \mathbb{C}\backslash_{\mathbb{R}^+}$. Thus, the Riemann surface consists of two copies of U glued along $\mathbb{R}^+$ (the *branch cut*).

Alternately, suppose f has power series g_0 at a point z_0. Take the collection $\{(z, g)\}$ of power series g around some z obtained by analytic continuation. By identifying (z, g_1) and (z, g_2) if $g_1 = g_2$ in a neighborhood of z, we obtain the Riemann surface on which the extension of f is well-defined.

One can also obtain a Riemann surface as some domain modulo a group of Möbius transformations acting discontinuously. For example, the quotient of $\mathbb{C}$ by a group of translations represents a torus.

9.2 Conformal Mapping Areas

dhh 4.10

For conformal maps, the image of a region R under f has area $\int\int_R |f'(z)|^2 dxdy$. This can be used to show that a power series f converging on the unit disk maps D to a region with area $\pi \sum k|a_k|^2$. Similarly, for $g(z) = z + \sum b_k z^{-k}$ converging outside the unit disk, the area omitted by g is $|\mathbb{C}\backslash_{g(\{|z|>1\})}| = \pi(1 - \sum k|b_k|^2)$.

9.3 Residue Theory and Calculus

For a two-variable rational function R, $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$ can be computed by substituting $z = \exp(i\theta)$ to obtain $\int_C R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{dz}{iz}$, which may be computed by the residue theorem.

If R is a rational function which disappears at ∞, then $\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\Im(b_i)>0} \mathrm{res}(R, b_i)$, summing over poles in the upper half plane. This is proven by using a contour $[-R, R] + C$, where C is a half circle connecting R to $-R$ in the upper half plane, and taking the limit. Then $\int_C R \to 0$, allowing the computation of the integral.

Similarly, for $a > 0$, $\int_{-\infty}^{\infty} R(x)e^{iax}dx = 2\pi i \sum_{\Im(b_i)>0} \mathrm{res}(R, b_i)e^{iab_i}$.

If R has a pole at a zero of $\sin(x)$ or $\cos(x)$, the integral may still be calculated. For example, $\int_{-\infty}^{\infty} \frac{sin(x)}{x}dx$ has a single pole at 0. To evaluate, we use the contour $[-R, -r] + C + [r, R] + D$ where C is a small semicircle connecting $-r$ to r, and D is a large semicircle connecting R to $-R$, both in the upper half plane.

Another type of integral is $\int_0^{\infty} R(x)f(x)dx$, where $zR(z)$ is a rational function disappearing at ∞ and either $f(x) = x^a$ with a a real constant or $f(x) = log(x)$. In the complex plane, $f(z)$ requires a branch cut, so a contour similar to the one above must be used; in this case, the whole branch cut must be avoided. Specifically, the contour is $[r + i\varepsilon, R + i\varepsilon] + C + [R - i\varepsilon, r - i\varepsilon] + D$, where $\varepsilon > 0$, and C, D are nearly whole circles connecting the endpoints of these lines, so that the positive real axis (the branch cut in this case) is avoided.

In a similar vein, exact answers for series, such as $\sum_{n=1}^{\infty} \frac{1}{n^k}$ can be computed using residue theory. The trick is to find a function with simple poles at the integers. Then, a suitable integral involving this function will evaluate to the sum of $\frac{1}{n^k}$ using residue theory, plus another term which involves the function evaluated at certain poles. If the whole integral goes to zero, then $\sum_{n=1}^{\infty} \frac{1}{n^k}$ may be calculated explicitly.