A Rough Guide to the Fundamental Group

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1 Getting Oriented

The fundamental group is a tool used to study topological spaces; its most important property is *topological invariance*, meaning that it is always the same for homeomorphic spaces. What this means is that it can distinguish between non-homeomorphic spaces.

The easiest way to think of the fundamental group is as the set of loops embedded in a space, or as the set of closed paths a creature in the space may traverse. There is a natural equivalence relation between these paths given by *homotopy*: two paths are equivalent if they can be 'smoothly deformed' into each other. Surprisingly, with this equivalence relation one can give the very large set of paths the structure of a group.

This group tells a lot about the structure of a topological space. For every space, there is a space with trivial fundamental group, called the *universal cover* which can be completely wrapped around it. The universal cover is intricately related to the fundamental group. We will see that the spaces it can 'wrap around' are in a 1:1 correspondence with normal subgroups of the fundamental group.

The final section is on the Seifert-Van Kampen Theorem, a calculational tool allowing us to calculate the fundamental groups of more complicated spaces. In particular, it will lead us to the classification of surfaces.

2 The Fundamental Group

2.1 Homotopies

Homotopy: a family of maps parametrized by the unit interval, i.e., $F : X \times I \to Y$, or $F_i : X \to Y$.

Then, $F_0(X)$ and $F_1(X)$ are *homotopic* maps, and we write $F_0 \simeq F_1$. The *homotopy class* $[f]$ of f is the set of maps homotopic to f. Two spaces are *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : X \to Y$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, and we write $X \simeq Y$. A space is *contractible* if it is homotopy equivalent to a 1-point space.

Deformation Retract: $A \subset X$ for which there is an $F : X \times I \to X$ with $F_0 = 1_X$, $F_1(X) \subset A$ and $F_1|_A = 1_A$. A is a *strong deformation retract* if $F_t|_A = 1_A$ for all t. In either case, $A \simeq X$.

Relative Homotopy: a homotopy $F : X \times I \to Y$ with $F_t|_A = 1_A$. We write $F_0 \simeq F_1$ rel A. If F and G are (relative) homotopies with $F_1 = G_0$ we can concatenate F and G to obtain the (relative) homotopy $F * G$.

An important class of homotopies are those rel $X_{01} = X \times \partial I$. Using the fact that $F : X \times I \to Y$ is $\simeq$ rel X_{01} to a reparametrization $F(x, \phi(t))$, we have:

1. $C * F \simeq F * C \simeq F$ rel X_{01} for a constant map C;
2. every F has an inverse F^{-1} with $F * F^{-1} \simeq C$ rel X_{01};
3. if $F_1, G_1 \simeq F_2, G_2$ then $F_1 * G_1 \simeq F_2 * G_2$ rel X_{01}.

In terms of homotopy classes, we have (1) $[e] * [f] = [f] * [e] = [f]$, (2) $[f] * [f^{-1}] = [e]$, and (3) $[f] * [g] = [fg]$ is well-defined.

For a mapping cylinder M_f, $r : M_f \to Y$ is a strong deformation retract, so we can replace $f : X \to Y$ with inclusion $X \hookrightarrow M_f$ (up to homotopy). Moreover, the homotopy type of M_f (or C_f) depends only on the homotopy class of f.

If, as well, $F(z_\alpha, t) = z_{\alpha_0} \forall z_\alpha \in Z$, we say that the homotopy *keeps Z fixed*, and we write $F : \sigma \simeq \tau$ rel Z. In the case that $Y = I$, we say that F is a *path-homotopy* and that σ and τ are *path-homotopic*. A map with $F(0, t) = x_0 \forall t \in I$ and $F(1, t) = x_1 \forall t \in I$ is said to *keep endpoints fixed*, and written by $\sigma \simeq \tau$ rel $(0, 1)$.

Given a space X, the space E is said to be a *covering space* of X if there is a map $p : X \to E$ which is continuous, and a local homeomorphism.

2.2 The Fundamental Group

> **Fundamental Group:** the homotopy group $\pi_1(X, \{x_0\})$, the homotopy classes of closed paths (loops) from the base point.

By considering a change of base point, it is clear that $\pi_1(X, \{x_0\}$ depends only on the path component of x_0. Since $\pi_1(X \times Y, \{(x_0, y_0)\} = \pi_1(X, \{x_0\}) \times \pi_1(Y, \{y_0\})$, we can assume the space is path connected and omit the base point.

> **Simply connected:** an arcwise-connected space with trivial fundamental group. Homotopy equivalent spaces have the same fundamental group. Thus, contractible spaces, which are homotopy equivalent to a point, are simply connected. Actually, any loop $S^1 \to X$ which is homotopically trivial extends to a map $S^1 \hookrightarrow D^2 \to X$.

3 Covering Spaces

3.1 Covering Maps

> **Covering Map:** a map $p : X \to Y$ (where X and Y are arcwise-connected, locally arcwise-connected, Hausdorff spaces), for which each point $y \in Y$ has a neighborhood U with inverse image $p^{-1}(U)$ consisting of disjoint sets U_α each homeomorphic with U by $p|_{U_\alpha}$. The number of points in the inverse image of a point is constant, and called the *number of sheets* of the covering.

The simplest example of a covering space is $\mathbb{R} \to S^1$ with $t \mapsto e^{2\pi i t}$ (an infinite sheeted covering). Similarly, the map $S^2 \to \mathbb{R}P^2$ from the sphere to the projective plane is a double covering.

> **Lifting Problem:** an important question for covering spaces: when does a map $f : W \to Y$ lift to a map $g : W \to X$ with $f = g \circ p$? The *path-lifting property* says that a path $f : I \to Y$ can be uniquely lifted to a path $g : I \to X$, and the *homotopy-lifting theorem* says that a homotopy $F : W \times I \to Y$ with partial lift $f : W \times \{0\} \to X$ can be lifted uniquely to a homotopy $G : W \times I \to X$. In general, a unique lift exists iff $f_\#(\pi_1(W)) \subset p_\#(\pi_1(X))$, where the maps are assumed to preserve base points.

> **Universal cover:** Of particular note is the monomorphism $p_\# : \pi_1(X) \to \pi_1(Y)$. In particular, a space is simply connected iff it has no nontrivial covers. With few conditions, every space has a unique simply connected covering space, called the *universal cover*.

3.2 Deck Transformations

Note that, given a covering map $p : X \to Y$, the group $\pi_1(Y, y_0)$ acts on the *fiber* $p^{-1}(y_0)$ as a group of permutations. The action is given by lifting a loop in $\pi_1(Y, y_0)$ to a path starting at some $x \in p^{-1}(y_0)$... the endpoint of this path is the result of the action.

> **Isotropy Subgroup:** if $J = \pi_1(Y, y_0)$, this is given by $J_{x_0} = \{\alpha \in J : x_0 \cdot \alpha = x_0\} = \Im\{p_\# : \pi_1(X, x_0) \to pi_1(Y, y_0)\}$. In this case, $\phi : J_{x_0}\mathcal{J} \to F$ is a bijection, so there is a 1:1 correspondence between right cosets $p_\#\pi_1(X, x_0)\pi_1(Y, y_0)$ and the fiber $p^{-1}(y_0)$. This in turn implies that the number of sheets of the covering map is precisely the index of $p_\#(\pi_1(X, x_0))$ in $\pi_1(Y, y0)$. In the specific case of the universal cover, the number of sheets is just the order of $\pi_1(Y, y_0)$.

> **Deck transformation:** given a covering map $p : X \to Y$, it is a homeomorphism $D : X \to X$ of the cover. Deck transformations form a group $\Delta = \Delta_p$ under map composition.

Now, it is natural to ask when we can consider the quotient X/Δ of X by deck transformations. One would expect to get Y. This particular case can be assured if the subgroup $p_\# \pi_1(X, x_0)$ is normal in $\pi_1(Y, y_0)$, or alternatively if Δ acts transitively on $p^{-1}(y_0)$, in which case the covering map is said to be *regular*. In this case, we have $\Delta \approx \pi_1(Y, y_0)/p_\# \pi_1(X, x_0)$. In particular, if X is simply connected (perhaps the universal cover), then $\Delta \approx \pi_1(Y, y_0)$.

> **Properly discontinuous action:** the condition necessary for a group G acting on X to give a regular covering map $p : X \to X/G$, in which case $\pi_1(X/G) \approx G$.

Covering spaces can be classified, up to equivalence, by the subgroups of $\pi_1(Y, y_0)$ (they are in a 1:1 correspondence). If a space is assumed to be *semilocally 1-connected* or *locally relatively simply connected* (meaning each point has a simply connected neighborhood), then every connected space has a universal cover, which can be thought of as the set of paths from a given point.

4 The Classification of Surfaces

4.1 The Seifert-Van Kampen Theorem

The Seifert-Van Kampen is a very important computational tool used for computing the fundamental group, and essentially relates a space to (smaller) portions of that space.

> **Seifert-Van Kampen Theorem:** if $X = U \cup V$, where U, V, and $U \cap V$ are nonempty and connected, then $pi_1(X) \approx \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$, with base point $x_0 \in U \cap V$. Here, $G_1 *_A G_2$ is a *free group with amalgamation*, meaning the group $G_1 * G_2$ with relations given by maps $\phi_1(a) = \phi_2(a)$, where $\phi_i : A \to G$.

So, for example, if $U \cap V$ is simply connected, then $\pi_1(X) \approx \pi_1(U) * \pi_1(V)$. Likewise, if V is simply connected, then $\pi_1(X) \approx \pi_1(U)/N$, where N is a subgroup of $pi_1(U)$ generated by the image of $\pi_1(U \cap V)$.

4.2 Surfaces

5 Going Further

5.1 Higher Homotopy Groups

Denote the set of homotopy classes of maps $X \to Y$ by $[X; Y]$, and of maps $(X, A) \to (Y, B)$ by $[X, A; Y, B]$.

> **Pointed Space:** a space with a specified base point, such as (X, x_0).
>
> **Homotopy Group:** Maps preserving base points form the group $[X; Y]_*$ so that $[SX; Y]_* \cong [X \times I, A; Y, \{y_0\}]$, where $SX = (X \times I)/(\{x_0\} \times I \cup X \times \partial I)$ is the *reduced suspension*. $[SX; Y]$ forms a group with operation being the composition of maps, called a homotopy group.
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> **nth Homotopy Group:** The most often used homotopy group, defined by $\pi_n(Y, \{y_0\}) = [S^n; Y]_*$, where S^n is the n-sphere, and can be thought of as the n-fold reduced suspension of $S^0 = \{0, 1\}$ with base point 0. An alternate definition would be $\pi_n(Y, \{y_0\} = [I^n, \partial I^n; Y, \{y_0\}]$, since S^n is formed from I^n by collapsing the boundary to a point.

The homotopy of spheres is most easily calculated. $\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_n(S^k) = 0$ for $n < k$. $\pi_n(S^1) = 0$ for $n > 1$, but $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_{n+1}(S^n) \cong \mathbb{Z}_2$ for $n > 2$. In general, the homotopy groups are *stable*, in the sense that $\pi_{n+k}(S^n)$ is independent of n for large n.

> **Functor:** the homotopy group is functorial. This means that a map $\phi \in [Y, W]_*$ induces a group homomorphism $\phi_\# : \pi_1(Y, \{y_0\}) \to \pi_1(W, \{w_0\})$, with $\psi_\# \circ \phi_\# = (\psi \circ \phi)_\#$ and $\mathrm{Id}_\# = \mathrm{Id}$. Moreover, if ψ and ϕ are homotopic, then $\psi_\# = \phi_\#$.

6 The Road Ahead

The fundamental group is just the introduction to the vast subject of algebraic topology. It is a rather intuitive concept, but can be very difficult to calculate. There is another topological invariant, called *homology*, which turns out to be easier. It is less intuitive, calculated in terms of boundaries and pieces of a space rather than something concrete. However, the groups one obtains are always abelian; in fact, the first homology group is the abelianization of the fundamental group.

Another vital aspect of modern algebraic topology is cohomology theory, which deals with maps from bits and pieces of a space into some nice group like $\mathbb{Z}$. It pairs up nicely with homology, meaning there is a natural correspondence (called duality) between the two.

A third piece of algebraic topology is higher homotopy theory, which generalizes the fundamental group to maps from spheres S^n into a space. It retains the calculational complexity of the fundamental group, and working with this complexity requires a very deep theory best approached after working through homology and cohomology theory.