Topological and Lie Groups

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1 Getting Oriented

The study of topological/Lie groups is all about combining two very different structures: a topological (for topological groups) or a differential (for Lie groups) structure and a group structure. The interplay between these two structures produces a strikingly rich theory, and involves a good deal of analysis, algebra, and geometry.

Geometry is most important when trying to write down the necessary conditions for a Lie group to have subgroups, factor groups, and so on. Algebra comes into play when considering the tangent space of a Lie group, which has the structure of a *Lie algebra*. Surprisingly, up to a few small conditions, Lie algebras are in direct correspondence with Lie groups. There is also a foundational theory allowing the complete classification of almost all Lie algebras.

One example of a Lie group is the circle (viewed as a subset of the complex plane, it is a group under multiplication); the sphere in general is also a Lie group. The most common ones are, however, matrix groups, which can be given a natural differentiable structure (viewing $M_n(\mathbb{R})$ as a subset of $\mathbb{R}^{n^2}$ for example). These are of primary importance in the theory, since every Lie group/Lie algebra is in fact isomorphic to a matrix group/algebra.

Beyond the basic Lie theory is *representation theory*, which deals with the homomorphisms of Lie groups/algebras into matrix groups. It turns out that the trace of this map, which is then a function from the Lie group/algebra to $\mathbb{R}$ or $\mathbb{C}$, is pretty much all that is necessary to classify the representation.

In the following, we will look at the relationship between Lie groups and Lie algebras, at the Peter-Weyl Theorem (generalizing Fourier analysis), and at the classification of Lie algebras.

2 Topological Groups

2.1 Definitions

A topological group combines the ideas of a topological space and a group, and so can be considered either as a group whose operations are continuous or a topological space with a group structure:

> **Topological Group:** a Hausdorff topological space G which also has a group structure G such that multiplication and inversion are continuous.

The definition also extends to allow subgroups and homomorphisms of topological groups, with the added condition of continuity.

The maps $G \to G$ given by $h \mapsto ghg^{-1}$ and $h \mapsto h^{-1}$, as well as the **left/right translation maps** $L_g(h) = gh$ and $R_g(h) = hg$ are homeomorphisms. So, by applying a translation we see that neighborhoods of a given point act like neighborhoods of the identity e. In fact, we can even work with **symmetric subsets** ($A \subset G$ with $A = A^{-1}$), since these form a neighborhood basis of e. This fact simplifies many proofs.

If a set H is a (normal) subgroup of G then its closure $\bar{H}$ is also a (normal) subgroup. If H is normal, the factor group G/H will be a Hausdorff space, with continuous projection map $\pi : G \to G/H$. Moreover, with the quotient topology $G/\bar{H}$ is also a topological group. Under certain circumstances, it is also true that the universal cover of a topological group can be given a group structure. These facts will carry over to the study of Lie groups.

2.2 Group Actions

A group action on a space X is essentially a group of maps $X \to X$:

> **Group Action:** a topological group G acts on a space X if there is a continuous map $G \times X \to X$ with $(gh)(x) = g(h(x))$ and $e(x) = x$.

A point $x \in X$ has **orbit** $G(x) = \{g(x)|g \in G\}$, and **isotropy/stability group** $G_x = \{g \in G|g(x) = x\}$. An action is **transitive** if there is only one orbit, and **effective** if the only $g \in G$ fixing all points in X is the identity. For example, if a group acts on itself, then at the identity e we have $G(e) = G = G_e$. Moreover, the action is both transitive and effective. If a compact topological group G acts on a Hausdorff space X, then the map $G/G_x \to G(x)$ with $gG_x \mapsto g(x)$, is a homeomorphism.

One important example of a group action is the fundamental group acting on a covering space $p : X \to Y$. The **monodromy action** is defined as the action of $\pi_1(Y, y_0)$ on the fiber $p^{-1}(y_0)$. It is transitive, and gives a lot of information about the fundamental groups of covering spaces. We can also consider the **group of deck transformations** Δ, that is the group of maps $D : X \to X$ with $p \circ D = p$. If the action is transitive, the covering space is said to be **regular** and we have an isomorphism $\Delta \approx \pi_1(Y, y_0)/p_{\#}\pi_1(X, x_0)$. Alternately, one can obtain a regular covering map $p : X \to X/G$ from a group G acting on X if the action is **properly discontinuous**, that is every point of X has some neighborhood U with $g(U)$ intersecting U only for $g = e$.

Another example is a **flow** on a manifold M, which is just a smooth action $\mathbb{R} \times M \to M$. A flow induces a vector field on M uniquely, and all vector fields induce a flow.

3 Lie Groups: Examples and Basic Constructions

3.1 Introduction

The set of isometries of a Riemannian manifold are almost always a very interesting construction. They give insight into the structure of the manifold, in a sense by identifying its 'symmetries.' The set of isometries has two important properties: it forms a group under composition, and also has a differentiable structure. These are actually the defining properties of a Lie group:

Lie Group: a topological group with a differentiable structure.

Therefore, a Lie group is a set with compatible group and differentiable manifold structures. This simple definition allows room for Lie groups to be studied as geometric/algebraic entities of their own in addition to their initial purpose as geometric tools for studying manifolds.

Just as Lie groups can be used to study manifolds, so *representation theory* can be used to study Lie groups. A **representation** of a Lie group G is basically just a homomorphism from G into another group, usually a group of matrices. Matrix groups themselves are of vital importance to Lie theory: from the algebraic point of view, Lie groups can be thought of as something slightly more general than matrix groups, as every matrix group is a Lie group. The simplest and most commonly arising Lie groups, usually called the "classical Lie groups," are all matrix groups, although they do have a geometric interpretation.

3.2 Geometric Examples

We will detail many common examples of Lie groups, including the classical Lie groups, in this section. As a first example, however, we have all finite groups, which are actually 0-dimensional compact Lie groups.

Following the geometric point of view, our first example of a Lie group is the translations along a line. As maps $f : \mathbb{R} \to \mathbb{R}$ we have the group $\{f(x) = x + a : a \in \mathbb{R}\}$. Thus, both the differential and group structure is that of $\mathbb{R}$ itself. Generalizing, we see that $\mathbb{R}^n$ is always a Lie group. Closely related is the group of *isometries* of $\mathbb{R}$: an isometry is just a translation or a reflection and translation, so the group is just two copies of $\mathbb{R}$, or $\mathbb{R} \times \mathbb{Z}_2$. Indeed, a Lie group need not be connected. The corresponding groups for the circle are the rotations (forming a group S^1) and the isometries (forming $S^1 \times \mathbb{Z}_2$). This is actually the **orthogonal group** $O(2)$ consisting of 2×2 orthogonal matrices, since such matrices preserve the unit circle. The line and the circle are the only 1-dimensional Lie groups, since they are the only manifolds available.

Moving up a dimension introduces many more manifolds, so it is best to just look at a couple of them. The linear transformations of the plane form $GL(2, \mathbb{R})$, the group of invertible 2×2 matrices. Isometries of the plane have the form $Ax + b$, where $A \in O(2)$ the orthogonal group from before. Thus, the isometry group is $(S^1 \times \mathbb{Z}_2) \times \mathbb{R}^2$. Again, there are two components, one consisting of those isometries *preserving* orientation, and one of those *reversing it* (a coset of the first). In general, the group of isometries of Euclidean space $\mathbb{R}^n$ is $O(n) \times \mathbb{R}^n$.

3.3 Lie Subgroups, Quotient Groups, and Covering Groups

This section describes some basic constructions on a Lie group. First, we have the *Lie subgroup.* This definition must be very precise, since a subset of a manifold may not have the same topology as a manifold. Therefore, we use two definitions: first, a **Lie subgroup** is a subgroup of G which is also a closed submanifold; second, a **immersed subgroup** is the image of a Lie group H under an injective morphism into G (and is not necessarily a closed submanifold).

For a quotient group $G = H/\Lambda$ to have a Lie group structure, one can show that Λ must be a discrete subgroup of the center $Z(H)$; the Lie group structure is unique, if the projection map $H \to G$ is to be compatible with the structures. Similarly, a covering map $\phi : H \to G$ gives a unique Lie group structure on H for every $e' \in \phi^{-1}(e)$ (element mapping to the identity), for which ϕ is a compatible map. In this case, the kernel of ϕ is exactly the center of H. This construction actually gives an important equivalence relation on Lie groups: G G' iff they have the same universal cover.

3.4 The Classical Lie Groups

We've already encountered some of the classical Lie groups: $GL(2, \mathbb{R})$ and $O(2)$. Here they are, along with some of their friends and what the letters stand for:

- $M(n)$ (**matrix group**): $n \times n$ matrices;
- $GL(n, \mathbb{R})$ (**general linear group**): nonsingular matrices;
- $SL(n, \mathbb{R})$ (**special linear group**): matrices with determinant 1;
- $O(n)$ (**orthogonal group**): matrices with $AA^t = I$;
- $SO(n)$ (**special orthogonal group**): matrices in $O(n)$ with determinant 1.

As indicated above, those with the clearest geometric interpretation are $O(n)$, the group of isometries of S^{n-1}, and $SO(n)$, the orientation-preserving isometries. As we might expect, we have $O(n) \cong SO(n) \times \mathbb{Z}_2$.

There are also the Lie groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $U(n)$, and $SU(n)$ corresponding to the complex case. Here, $U(n)$ is the **unitary group**, consisting of matrices with $AA^* = I$, A^* being the conjugate transpose. Prefixing the S indicates, as usual, that the subgroup of matrices with determinant 1.

There are also corresponding groups for the *quaternions* $\mathbb{H}$, denoted $GL(n, \mathbb{H})$, $SL(n, \mathbb{H})$, and $Sp(n)$. This last is the **symplectic group**. That these three families are the most important and most studied Lie groups follows from the *Frobenius Theorem*, which essentially says that the only finite-dimensional real division algebras are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

3.5 Homogeneous Spaces

The desire to give a Lie group a geometric interpretation gives rise to the somewhat reverse-logical notion of a **homogeneous space** (for a Lie group G), which is just a space on which G acts transitively, i.e., nicely. Homogeneous spaces are very important in the study of Lie groups, and give meaning to the 'tool' characterization: a homogeneous space is one which can be studied with a Lie group 'tool'.

We have already looked at the most basic example: the underlying space of a group of isometries, like $O(n)$. Another example of this type is the upper half-plane $\mathbb{H}$ of $\mathbb{C}$, on which $SL(2, \mathbb{R})$ acts by $z \mapsto \frac{az+b}{cz+d}$ (these are all Möbius transformations). An example of a different type is the space $\mathcal{L}$ of lattices in $\mathbb{R}^2$, which is homogeneous under $GL(2, \mathbb{R})$. Moreover, the space of *unimodular* lattices, those whose basic parallelogram has unit areas, is homogeneous under $SL(2, \mathbb{R})$.

4 Lie Groups and Lie Algebras

A Lie algebra can be described as the 'linearization' of a Lie group. It is the first link in a chain of simplifications: one can then pass to a complex Lie algebra (by tensoring with $\mathbb{C}$), and then to a *semisimple* complex Lie algebra, and finally to a *simple* complex Lie algebra. Actually, one could even regard passing from a manifold to its transformation group (a Lie group) as the first link in the chain and so relate manifolds and simple complex Lie algebras. The important thing to keep in mind here is that the Lie algebra is just another tool: you bring it when you want to eliminate the topological complexity of the Lie group.

4.1 Lie Algebras

Lie algebras, like Lie groups, can be introduced either geometrically or algebraically. Here is the geometric description:

Lie algebra: the tangent space of a Lie group at its identity

Studying the Lie algebra eliminates the topological complexity of a Lie group, while keeping its algebraic structure. Indeed, by applying a left translation, one sees that the the tangent space to an arbitrary element $x \in G$ acts just like the tangent space at the identity.

For essentially the same reason, any map of Lie groups $\rho : G \to H$ is completely determined by its differential $d\rho_e : T_eG \to T_eH$ at the identity. It is natural to ask: what maps $T_eG \to T_eH$ arise as the differential of a Lie group map? The answer gives the defining quality of a Lie algebra: those maps which preserve a certain *bracket operation*. The properties of the bracket give the algebraic definition of a Lie algebra:

Lie algebra: a vector space g with a skew-symmetric bilinear map $\mathsf{g} \times \mathsf{g} \to \mathsf{g}$ denoted $[\cdot, \cdot]$ satisfying the Jacobi identity. Thus, $[X, Y] = [Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

4.2 The Bracket Operation

The bracket operation given above can be written down explicitly in terms of the operations on G, but requires a few intermediate steps. Our guiding question, stated above, is to characterize in terms of differentials which maps $G \to H$ are Lie group maps.

First, a Lie group map $\rho : G \to H$ must preserve the composition operation $\Psi : G \to \mathrm{Aut}(G)$ given by $\Psi_g(h) = ghg^{-1}$. This is trivial: ρ is a homomorphism, so all we're saying is that $\rho \circ \Psi = \Psi \circ \rho$, or $\rho(ghg^{-1}) = \rho(g)\rho(h)\rho(g)^{-1}$.

Now, we're concerned with the differential, so: if ρ is a Lie group map then $d\rho_e$ respects the action of $(d\Psi_g)_e$ on the tangent space at the identity T_eG. The map $d(\Psi_g)_e$ is called the **Adjoint map** or **adjoint representation** and denoted $\mathrm{Ad}_g \in \mathrm{Aut}(T_eG)$. So for a Lie group map ρ we have $d\rho(\mathrm{Ad}_g(v)) = \mathrm{Ad}(\rho(g))d\rho(v))$ for $v \in T_eG$.

This last condition still depends on ρ as well as $d\rho$, so we have one more step: take another differential. We have $\mathrm{ad} = d(\mathrm{Ad}) : T_eG \to \mathrm{End}(T_eG)$, called the **adjoint map**. Thus, every ad_X is a map $T_eG \to T_eG$. Alternately, one may regard ad as a function of two variables, giving a map $T_eG \times T_eG \to T_eG$. This last characterization allows us to define the **bracket operation** by $[X, Y] = \mathrm{ad}_X(Y)$. Now the condition for a map ρ to be a Lie group map becomes $d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]$, so the differentials of such maps preserve the bracket operation.

As stated above, the bracket characterizes a general Lie algebra. The fact that it is skew-symmetric and satisfies the Jacobi identified may be verified using the properties of the adjoint representation. Any time a Lie group is given as a subgroup of the general linear group $GL_n(\mathbb{R})$, the bracket coincides with the commutator $X \cdot Y - Y \cdot X$ of the Lie algebra (which is a subset of $\mathrm{End}(\mathbb{R}^n)$). This gives an intuitive feeling for the bracket. However, in general the operation $X \cdot Y$ may not exist, so one must rely on the definition in terms of the adjoint representation.

4.3 Matching Lie Groups to Lie Algebras

One can also define a **Lie algebra representation** as a Lie algebra map $\rho : \mathsf{g} \to gl(V) = \mathrm{End}(V)$ (thus preserving brackets). With this definition in place, we have four basic but paramount facts regarding a Lie group and its Lie algebra:

- the tangent space at the identity of a Lie group has a natural Lie algebra structure;
- the Lie group maps $G \to H$ are in 1:1 correspondence (when G is connected/simply-connected) with the associated Lie algebra maps $\mathsf{g} \to \mathsf{h}$ (given by taking the differential);
- the representations of a connected/simply-connected Lie group are in 1:1 correspondence with the representations of its Lie algebra (given by taking the differential);
- connected/simply-connected Lie groups are in 1:1 correspondence with Lie algebras.

This last statement gives life to the equivalence relation we gave above: Lie groups with the same universal cover are those with the same Lie algebra.

4.4 The Exponential Map

We should mention a few things about the exponential map $\mathsf{g} \to G$, since it explicitly demonstrates how the algebraic structure of a Lie group is still there in its Lie algebra. There are a number of ways to see how this map is defined. Intuitively, it describes how to 'wrap' the tangent space around the manifold.

To give a precise definition, we return to the idea of translating by group elements. Each element $X \in \mathsf{g}$ corresponds to a unique left-invariant vector field on G by translating X by g. Then, one has a **1-parameter subgroup of** G defined by the integral curve starting at $e \in G$ and tangent to the translates of X. This is analogous to following a geodesic path on a Riemannian manifold, basically tracing out a 'straight' line from e in the direction X.

The difference for Lie groups is that the integral curve $\varphi_X : U \to G$ tangent to a left-invariant vector field with U a neighborhood of 0 and $\varphi_X(0) = e$ and $d\varphi_X(0) = X$ turns out to be a homomorphism: $\varphi_X(s + t) = \varphi_X(s) \cdot \varphi_X(t)$. The **exponential map** $\exp : \mathsf{g} \to G$ takes the vector $X \in \mathsf{g}$ to $\varphi_X(1)$. Since $\varphi_{\lambda X}(t) = \varphi_X(\lambda t)$, it can be defined on all of g. Clearly, it takes 0 to e and its differential at the origin is the identity. The exponential map is an isomorphism in the neighborhood of the identity, and encodes the algebraic structure of the Lie group by the *Campbell-Hausdorff Formula*:

$$X * Y = \log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] \pm \frac{1}{12}[X, [X, Y]] \pm \frac{1}{12}[Y, [Y, X]] + \cdots .$$

We can also now write down the correspondence between subgroups and subalgebras: if $\mathsf{h} \subset \mathsf{g}$ is a Lie subalgebra, then $\exp(\mathsf{h})$ is an immersed subgroup of G, with tangent space h.

4.5 The Classification of Lie Algebras

Because of the close correspondence between Lie groups and Lie algebras, it is natural to consider the classification of Lie algebras. We do not go into the details here, but the class of *simple* Lie algebras, those with no nontrivial ideals (subalgebras), can be completely classified. This involves looking at the *Cartan decomposition* of the Lie algebra corresponding to a subalgebra which is maximal in a certain sense (called the *Cartan subalgebra*). One then looks at the configuration of the basis elements of this decomposition (called the *roots*). One can enumerate all possible configurations of roots using graph theory and *Dynkin diagrams*, and then translate these back into algebraic language to give a few infinite classes of simple Lie algebras plus a few extras.

We will actually see a similar decomposition when we look at the maximal torus of a Lie group. The resemblance is not a coincidence; rather, it corresponds to the natural correspondence of Lie groups and Lie algebras.

5 Representations of Lie Groups; the Peter-Weyl Theorem

We described representations in the introduction as homomorphisms from a Lie group into some other group. Thus, one may or may not lose some of the structure of the group, much like taking a photograph of a 3-dimensional object. (If none is lost, the representation is said to be *faithful*).

5.1 Fourier Series

Fourier series are an example of a representation of the Lie group S^1, and as such provide much of the impetus for representation theory. The decomposition of an arbitrary continuous function on S^1 into exponential functions generalizes to all compact Lie groups, the content of the *Peter-Weyl Theorem*.

5.2 The Peter-Weyl Theorem

6 The Road Ahead