A Rough Guide to Point-Set Topology

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1 Getting Oriented

Point-set topology is the study of the intrinsic properties of surfaces which are *not* related to distance. The classic example is the donut and the coffee cup, which, from our point of view, will be the same object.

We begin by looking at *metric spaces*, for which distance is defined, allowing the definition of an *open sets* as a bunch of points which are "close together". Then we will take away the metric (distance), and just look at a set of points. How can we tell how those points are connected without distance? Well, we just skip the metric and begin by defining which sets of points are open (close together). This collection of open sets is what we mean by a *topology*.

Perhaps the most important concept here is the notion of *homeomorphic* topological spaces, which are just those that have the same topology... the underlying points are connected in the same manner (back to the donut and the coffee cup). This concept is representative of much of mathematics. There are many different ways to define how sets connect, and most of these give rise to notions of equivalence between sets. This naturally leads to a general question which is responsible for much of today's mathematics: how many different objects are there?

2 The Basics

2.1 Metric Spaces

A metric space is a set of points equipped with a notion of distance:

Metric Space:

a pair (X, d) with $d : X \times X \to \mathbb{R}$ such that:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (positivity);
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The basic open set in X is the **ϵ-ball** $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$. A metric space is **totally bounded** if $\forall \epsilon > 0$ it can be covered by a finite number of ϵ-balls.

A sequence in X is **Cauchy** if given $\epsilon > 0$ there is an $N > 0$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$. **Complete** spaces are those in which every Cauchy sequence converges.

2.2 Topological Spaces

As stated in the introduction, a topological space is a set of points equipped with a notion of "closeness":

Topological Space:

a pair (X, τ), where τ consists of all open sets in X and satisfies:

1. the union of open sets is open;
2. the finite intersection of open sets is open;
3. the sets X and $\emptyset$ are open.

The collection τ of subsets is known as a **topology** for X, and precisely defines the open sets (those in τ) and closed sets (those whose complement is in τ). A **small/weak/coarse** topology has a very small number of open sets, while a **large/strong/fine** topology has a very large number.

Bases for a Topology

When it's not practical to give a list of all open sets, one might use a basis or subbasis for the topology. A **basis** is a collection of open sets whose unions give the topology, while a **subbasis** is a collection of open sets whose unions and finite intersections give the topology.

A **neighborhood basis** around a point x is a collection of neighborhoods $\{B_i\}$ of that point such that any neighborhood of x is contained in some B_i. X is **First Countable** if every point has a countable neighborhood basis, and **Second Countable** if the topology has a countable basis.

Subsets

Subsets $A \subset X$ inherit a topology from X. The **closure** $\overline{A}$ of a set $A \subset X$ is the set of limit points of A, or equivalently the intersection of all closed sets containing A. The **interior** A° is the union of open subsets of A. A is **dense** if $\overline{A} = X$, and **nowhere dense** if $\overline{A}^\circ = \emptyset$.

Connected space:

a space X which is not the disjoint union of two nonempty open subsets. Equivalently we may require (i) the only *clopen* (both closed and open) sets are X and $\emptyset$, or (ii) every discrete valued map on X is constant.

The **components** of X are the largest connected subsets in X, and the subtler **quasi-components** are given by the equivalence relation $p \sim q$ when $v(p) = v(q)$ for all discrete valued maps v on X.

Maps

A map $f : X \to Y$ is **continuous** if $invf(U)$ is open whenever $U \subset Y$ is open. It is **continuous at a point** x if it is continuous on a neighborhood of x. f is **open (closed)** if $A \subset X$ open implies $f(A)$ is open (closed). Topological spaces are equivalent if one can find a continuous map between them with continuous inverse:

Homeomorphism:

a bijective map f with both f and f^{-1} continuous.

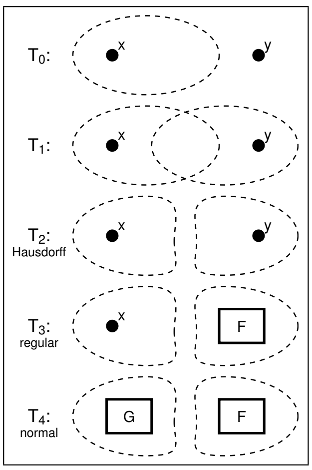
3 Some Important Types of Topological Spaces

3.1 The Separation Axioms

The *separation axioms* (denoted T_i) classify X based on how 'easy' it is to separate distinct points (or closed sets) in X. The following are the conditions holding in each case:

- T_0: there is an open set containing only one of the points;
- T_1: there are open sets about each of the points not containing the other;
- (**Hausdorff**) T_2: T_1 with the open sets disjoint;
- (**regular**) T_3: there are disjoint open sets for any point x and closed set F;
- (**completely regular**) $T_{3.5}$: there is an $f : X \to [0, 1]$ with $f(x) = 0, f(F) = 1$;
- (**normal**) T_4: there are disjoint open sets for any two closed sets.

The Hausdorff condition is probably the most important, since all of differential geometry assumes this condition. The normal condition is also important, since it is the characterization of metric spaces.



Topological Separability

Some Important Theorems

The stronger conditions here assert the existence of functions with special properties: the **Urysohn Metrization Theorem** states that a 2nd countable/completely regular space can be given a metric; **Urysohn's Lemma** states that in a normal space, for any two closed/disjoint sets C, D there is a function $f : X \to [0, 1]$ with $f(C) = 0$ and $f(D) = 1$; and the **Tietze Extension Theorem** states that a continuous map $f : C \to \mathbb{R}$ on a closed subset C of a normal space can be extended to a map $\hat{f} : X \to \mathbb{R}$.

3.2 Compactness

Compact spaces are the "small" ones:

Compact space:

a space X for which every open cover has a finite subcover. Equivalently, if all finite subcollections of some collection of sets have nonempty intersection, the entire collection has nonempty intersection.

Closed subsets of a compact set are compact. If $f^{-1}(C)$ is compact for all compact C, the map f is called **proper**. The following properties of compact spaces are very useful:

- a closed subset of a Hausdorff space is compact; a compact/Hausdorff space is normal;
- a real-valued map on a compact space assumes a maximum value;
- a metric space is compact iff every sequence has a convergent subsequence iff it is complete/totally bounded;
- (**Lebesgue Lemma**): for any open cover of a compact metric space, $\exists \delta > 0$ such that any set of diameter $< \delta$ is contained in some set of the cover.

Local Compactness

A space is **locally compact** if every point has a compact neighborhood. A locally compact/Hausdorff space is completely regular. On such a space, one can define the **1-point compactification** $X^+ = X \cup \{\infty\}$ by adjoining a **point at infinity** and defining $(X \setminus C) \cup \{\infty\}$ to be open for all compact C.

Proper maps on locally compact/Hausdorff spaces are closed, and extend to continuous maps on the 1-point compactification. A space X is completely regular iff it can be embedded as a subspace in a compact/Hausdorff space.

By the **Baire Category Theorem**, when X is a complete metric space or a locally compact Hausdorff space, the union of a countable number of nowhere dense subsets of X is nowhere dense; equivalently, the intersection of a countable family of dense open sets is dense.

Paracompactness

A space is **paracompact** if it is Hausdorff and every open cover has a locally finite refinement, i.e., a subcover meeting each point in a finite number of sets. If X is paracompact with open cover $\{U_\alpha\}$, then (i) $\{U_\alpha\}$ has a subordinate cover with a partition of unity; and (ii) there is an open cover $\{V_\alpha\}$ with $\overline{V_\alpha} \subset U_\alpha$. Moreover, a paracompact space is normal, and a second countable space is paracompact iff it is a metric space.

σ-Compactness

A space is **σ-compact** if it is the union of countably many compact subspaces. A locally compact space is paracompact iff it is the disjoint union of open σ-compact subsets. A locally compact/Hausdorff/second countable space X is σ-compact and paracompact, and X^+ is then metrizable.

4 Creating New Spaces: Products and Quotients

4.1 The Product Topology

A basis for a topology can be defined on the Cartesian product of several topological spaces:

Product Topology:

in the product space $\prod X_\alpha$, a set $\prod U_\alpha$ is defined to be open iff each $U_\alpha \subset X_\alpha$ is open and the set $\{U_\alpha \neq X_\alpha\}$ is finite.

If $X_\alpha = X$ for $\alpha \in A$, the product space is X^A, the space of functions $A \to X$. One may also define a topology (the obvious one) on a disjoint union $X \sqcup Y$.

4.2 The Quotient Topology

Given a surjective map $f : X \to Y$, one can define a topology on Y by letting $V \subset Y$ be open iff $f^{-1}(V)$ is open in X. This is precisely the topology which makes f continuous, called the **quotient topology**. A related notion is the quotient space:

Quotient space:

the space $(X/\sim)$ with topology given by the inclusion $(X/\sim) \hookrightarrow X$, where $\sim$ is an equivalence relation on X. A subcase is X/A, meaning the set A is collapsed to a point.

If A is closed, then X regular (normal) $\implies$ X/A Hausdorff (normal). Here are some important quotient spaces:

- **identification space**: for $f : A \subset Y \to X$, it is $X \cup_f Y = X \cup Y/(a \sim f(a))$;
- **retract**: $A \subset X$ with **retraction** $f : X \to A \subset X$ fixing each point in A;
- **mapping cylinder**: $M_f = Y \cup_f (X \times I)$ where $f : X \times \{0\} \to Y$;
- **mapping cone**: $C_f = M_f/X \times \{1\}$ (so the top collapses to a point).

A function on a mapping cylinder is continuous iff the induced functions on $X \times I$ and Y are continuous.

5 Going Further

5.1 Homotopy

One notion of equivalence between maps is given by homotopy, and can be thought of as a way to deform one map into another:

Homotopy:

a family of maps parametrized by the unit interval, i.e., $F : X \times I \to Y$, or $F_i : X \to Y$. $F_0(X)$ and $F_1(X)$ are said to be **homotopic** maps, and we write $F_0 \simeq F_1$. The **homotopy class** $[f]$ of f is the set of maps homotopic to f.

Note that we can concatenate homotopies F and G if $F_1 = G_0$. The resulting homotopy is denoted $F * G$.

Homotopy Equivalence

Two spaces are **homotopy equivalent** if there exist maps $f : X \to Y$ and $g : X \to Y$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, and we write $X \simeq Y$.

A space is **contractible** if it is homotopy equivalent to a 1-point space. A subset $A \subset X$ is a **deformation retract** if there is a homotopy F with $F_0 = 1_X$, $F_1(X) \subset A$ and $F_1|_A = 1_A$. A is a **strong deformation retract** if $F_t|_A = 1_A$ for all t. In either case, $A \simeq X$.

Relative Homotopy

A **relative homotopy** stays fixed on some set, i.e., $F_t|_A = 1_A$. We write $F_0 \simeq F_1$ rel A. Homotopies fixed on $X_{01} = X \times \partial I$ are very important. Using the fact that $F : X \times I \to Y$ is $\simeq$ rel X_{01} to a reparametrization $F(x, \phi(t))$, we have:

1. $C * F \simeq F * C \simeq F$ rel X_{01} for a constant map C;
2. every F has an inverse F^{-1} with $F * F^{-1} \simeq X_{01}$;
3. if $F_1, G_1 \simeq F_2, G_2$ then $F_1 * G_1 \simeq F_2 * G_2$ rel X_{01}.

So we have $[e] * [f] = [f] * [e] = [f]$; $[f] * [f^{-1}] = [e]$; and $[f] * [g] = [fg]$ is well-defined. In other words, such homotopies form a group. This is the starting point for the fundamental group and algebraic topology.

6 The Road Ahead

The natural follow-up to point-set topology is algebraic topology, which usually begins with the study of homotopy classes of maps. Specifically, the set of maps from the circle S^1 into a topological space forms a group called the *fundamental group* of the space. This group is unchanged by homeomorphism, and so can be used to distinguish and classify topological spaces.