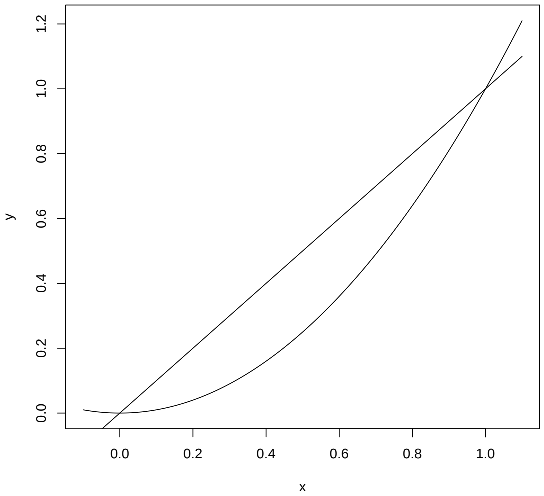
MATH 141 **EXAM 1 SOLUTIONS**

1. (a) Sketch of R, the region bounded by the curves $y = x$ and $y = x^2$.



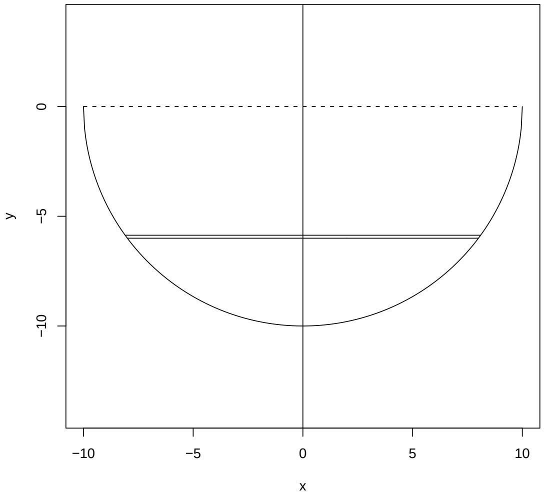
(b) The region is rotated about the x-axis. What is the volume of the resulting solid of rotation?

$$\begin{aligned} V &= \int_0^1 \pi[x^2 - (x^2)^2]dx \\ &= \pi[x^3/3 - x^5/5]\Big|_0^1 \\ &= 2\pi/15. \end{aligned}$$

(c) The lower boundary of R is the segment of the parabola $y = x^2$ between $x = 0$ and $x = 1$. Its length is the following integral:

$$L = \int_0^1 \sqrt{1 + (2x)^2}dx.$$

2. A hemispherical tank is filled with a liquid which weighs 40 lb/cu.ft. The radius of the tank is 10 ft. All of the liquid is pumped out of the tank from the very top. A sketch of the vertical cross-section of the tank is given below.



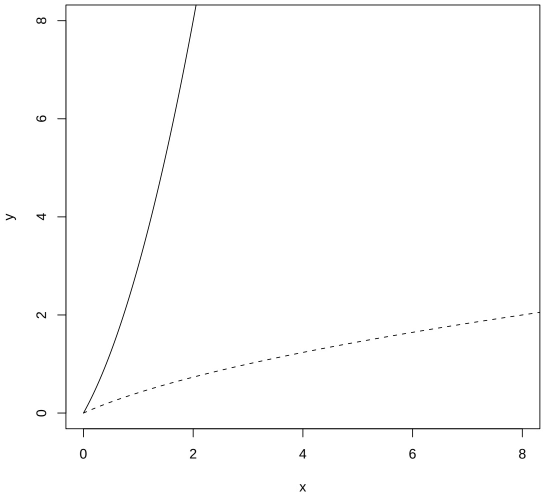
From the diagram, one can see that the points $(0, y)$, (x, y) and $(0, 0)$ form a right triangle with height $|y|$, base x and hypotenuse 10. The area of the circular cross section of the thin slab of liquid at level y is $\pi x^2 = \pi(10^2 - y^2)$ and its weight is $40\pi(10^2 - y^2)dy$. This thin slab must be raised a distance $-y$. Therefore the work against the force of gravity to empty the tank is as follows.

$$\begin{aligned} W &= \int_{-10}^{0} F(y)(-y)\,dy = \int_{-10}^{0} 40\pi(10^2 - y^2)(-y)\,dy \\ &= 40\pi\left[\frac{y^4}{4} - \frac{100y^2}{2}\right]_{-10}^{0} = 100,000\pi \end{aligned}$$

3. Let $f(x) = x^2 + 2x$ when $0 \le x < \infty$.

(a) To show that f^{-1} exists, note that $f'(x) = 2x + 2 \ge 0$ whenever $x \ge 0$. Therefore $f(x)$ is strictly increasing and invertible. One could also solve $x = f(y) = y^2 + 2y$ to obtain $y = f^{-1}(x) = -1 + \sqrt{1+x}$. This is the unique solution for $x \ge 0$.

(b) The graphs of f and f^{-1} appear below. The solid line is the graph of f and the dashed line is the graph of f^{-1}.



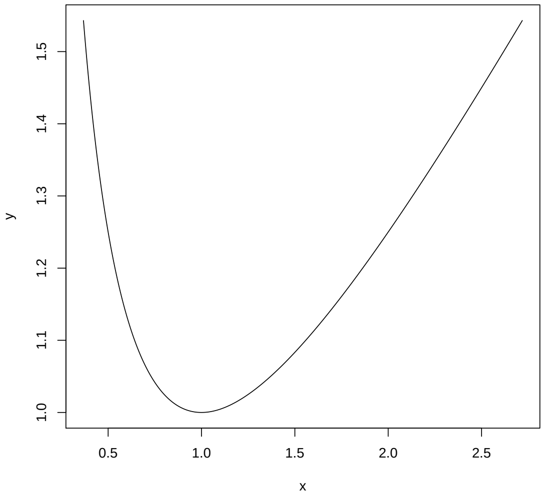
(c) To find $(f^{-1})'(2) = (d/dx)f^{-1}(x)|_{x=2}$ we need $(f^{-1})(2)$. Therefore we must solve the quadratic equation $f(x) = x^2 + 2x = 2$. The solution is $\sqrt{3} - 1$ (we only want a nonnegative solution). Then

$$\frac{d}{dx}f^{-1}(x)\bigg|_{x=2} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{2(\sqrt{3}-1)+2} = \frac{1}{2\sqrt{3}}.$$

4. The curve C is parameterized by $x = e^t$, $y = (1/2)(e^t + e^{-t})$ for all t.

(a) To reexpress C by a single equation involving x and y, substitute $x = e^t$ into the equation for y to obtain $y = (1/2)(x + 1/x)$.

(b) The sketch of C appears below. As $t \to -\infty$, $x \to 0$ and $y \to +\infty$. As $t \to +\infty$, $x \to +\infty$ and $y \to \infty$. The direction of increase in t is the same as the direction of increase in x.



(c) Write an integral representing the length of the curve from $t = -1$ to $t = 1$. Do **not** evaluate this integral.

$$\begin{aligned} L &= \int_{-1}^{1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_{-1}^{1} \sqrt{e^{2t} + [(1/2)(e^t - e^{-1})^2}\, dt = \int_{-1}^{1} (1/2)\sqrt{5e^{2t} - 2 + e^{-2t}}\, dt \\ &= \int_{e^{-1}}^{e} \sqrt{1 + [(1/2)(1 - 1/x^2)]^2}\, dx = \int_{e^{-1}}^{e} (1/2)\sqrt{5 - 2/x^2 + 1/x^4}\, dx. \end{aligned}$$