

1. Compute (by hand) the Cholesky factorization of

$$A = \begin{pmatrix} 4 & 6 & 10 \\ 6 & 25 & 19 \\ 10 & 19 & 62 \end{pmatrix}$$

2. Ex.2.11, p.87, *Numerical Computing with MATLAB*.
3. Consider the linear system

$$\begin{aligned} 6x_1 + 2x_2 + x_3 &= -3 \\ 2x_1 + \frac{2}{3}x_2 - \frac{4}{3}x_3 &= 4 \\ x_1 + 2x_2 - 3x_3 &= 2 \end{aligned}$$

- (a) Verify that its solution is

$$x_1 = 1.4 \quad x_2 = -4.2 \quad x_3 = -3.0$$

- (b) Using four digit floating-point decimal arithmetic with rounding, solve the system by Gaussian elimination without pivoting.
 - (c) Repeat part (b) using partial pivoting. In performing the arithmetic operations, remember to round to four significant digits after each operation, just as would be done on a computer. If you are careful you should see a significant difference.
4. The Hilbert matrix of order n , H_n is defined by

$$(H_n)_{i,j} = \frac{1}{i+j-1}, \quad i = 1, \dots, n, \quad j = 1, \dots, n.$$

H_n is nonsingular. However, as n increases, the condition number of H_n increases rapidly. H_n is a library function in MATLAB, `hilb(n)`. Let $n = 10$, $\mathbf{x} = \text{ones}(10, 1)$ and $\mathbf{b} = H_{10}\mathbf{x}$. Now use the backslash operator to solve the system $H_n\mathbf{x} = \mathbf{b}$, obtaining \mathbf{x}^* . Since we know \mathbf{x} exactly, we can compute $\mathbf{e} = \mathbf{x} - \mathbf{x}^*$, the error, and $\mathbf{r} = \mathbf{b} - H_{10}\mathbf{x}^*$, the residual. Compute these quantities and also `cond(H_n)` (a MATLAB function). Show that the two basic principles of solving linear systems by G.E./P.P. in floating point arithmetic hold. How many correct digits does \mathbf{x}^* have? Repeat with $n = 11, 12, \dots$. Stop when some component of \mathbf{x}^* has no correct digits.

5. Define the matrix A_n of order n by

$$a_{i,j} = \begin{cases} -1, & i < j \\ 1, & i = j \\ 0, & i > j \end{cases}$$

- (a) Show how to generate A_n in MATLAB with **eye**, **ones** and **triu**.
 (b) Find the inverse of A_n explicitly.
 Hint: Find the inverse of A_6 by using MATLAB. Then use the result to “guess” the inverse of A_n in general.
 In parts (c) and (d) use the ∞ -norm.
 (c) Show $\text{cond}(A_n) = n2^{n-1}$. For what n is this greater than $1/\epsilon$?
 (d) With $\mathbf{b} = [-n + 2, -n + 3, \dots, -1, 0, 1]^T$, the solution of $A_n \mathbf{x} = \mathbf{b}$ is $\mathbf{x} = [1, 1, \dots, 1]^T$. Perturb \mathbf{b} to $\hat{\mathbf{b}} = \mathbf{b} + [0, \dots, 0, \epsilon]^T$. Solve for $\hat{\mathbf{x}}$ in $A_n \hat{\mathbf{x}} = \hat{\mathbf{b}}$. Compare the relative error in $\hat{\mathbf{x}}$ to the relative error in $\hat{\mathbf{b}}$. What do you conclude?
 Hint $\hat{\mathbf{x}} = \mathbf{x} + A_n^{-1}(\hat{\mathbf{b}} - \mathbf{b})$.

6. Suppose \mathbf{x} satisfies $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} + \Delta\mathbf{x}$ satisfies $(A + \Delta A)(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$. Then we have the *condition number inequality*: If $\rho = \|A^{-1}\| \cdot \|\Delta A\| < 1$

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\text{cond}(A)}{1 - \rho} \left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} \right). \quad (1)$$

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} .5055 & .6412 & .8035 \\ .1693 & .0162 & .6978 \\ .5247 & .8369 & .4619 \end{pmatrix}, \quad \mathbf{b} = \begin{bmatrix} .0826 \\ .8207 \\ .1930 \end{bmatrix}$$

- (a) Solve $A\mathbf{x} = \mathbf{b}$ using the backslash operator.
 (b) Use equation (1) to answer the following question: If each entry in A and \mathbf{b} might have an error of $\pm .00005$, how reliable is \mathbf{x} ? Use the ∞ -norm.
 (c) Let
 $\Delta A = .0001 * \text{rand}(3) - .00005 * \text{ones}(3)$, $\Delta \mathbf{b} = .0001 * \text{rand}(3, 1) - .00005 * \text{ones}(3, 1)$.
 Solve $(A + \Delta A)(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$ to get $\mathbf{x} + \Delta\mathbf{x}$. Calculate $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$. Is this consistent with (b)? What is the relative change in each x_i ?

7.

- (a) Let A be an $n \times n$ matrix and $\mathbf{x} \in \mathbf{R}^n$. How many *flops* does it take to form the product $A\mathbf{x}$?
 (b) Let A and B be $n \times n$ matrices. How many *flops* does it take to form the product AB ?
 (c) In light of the results of (a) and (b), from the standpoint of efficiency, how should one compute $A^k \mathbf{x}$ for k a positive integer $k > 1$?

8. The solution of the boundary value problem

$$-\frac{d^2 u}{dx^2} + \frac{1}{x^2} u = -\frac{1}{x^3}, \quad u(1) = 1, \quad u(2) = \frac{1}{2}$$

is $u(x) = 1/x$. We wish to construct finite difference approximations to $u(x)$ as we did in class.

- (a) Use Moler’s fast diagonal solver **tridisolve** to find solutions to the system of finite difference equations for $n = 9, 19, 39$.
 (b) Evaluate the accuracy of your solutions by taking the norm of the difference of the solution vector and the vector of exact solutions values at the nodes.