MATH 240 **FINAL EXAM** **May 16, 2002**

Instructions: Number the answer sheets from 1 to 5. Fill out all the information at the top of each sheet. Answer problem n on page n, $n = 1, \cdots, 5$. Do not answer one question on more than one sheet. If you need more space use the back of the correct sheet.

SHOW ALL WORK

1. (45 points)

(a) (20 points) Let A be an $n \times n$ matrix. Complete the statements below in such a way that each statement is equivalent to the statement "A is an invertible matrix". In parts (v) and (vi) choose the word or words which complete the statement correctly.

(i) The columns of A are $\cdots$.

(ii) The rank of A is $\cdots$.

(iii) The determinant of A is $\cdots$.

(iv) The equation $A\mathbf{x} = \mathbf{0} \cdots$.

(v) $\lambda = 0$ (is, is not) an eigenvalue of A.

(vi) A (is, is not) row equivalent to I.

(b) (25 points) Let

$$A = \begin{pmatrix} 1 & 2 & 2 & -2 & 3 \\ 2 & 4 & 5 & -7 & 6 \\ 4 & 8 & 9 & -11 & 13 \\ 5 & 10 & 11 & -13 & 15 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrices A and B are row equivalent. Find

(i) The rank of A.

(ii) The dimension of the nullspace of A.

(iii) A basis for the column space of A.

(iv) A basis for the row space of A.

(v) A basis for the Nullspace of A.

2. (45 points)

(a) (25 points)

(i) Complete the following definition: Let V and W be vector spaces. A mapping $T : V \to W$ is a *linear transformation* if ______________

(ii) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation such that

$$T(2, 1)^T = (1, -1, 1)^T, \quad T(5, 3)^T = (2, 1, 4)^T$$

Find $T(1, 0)^T$ and $T(0, 1)^T$.

(iii) Find M, the matrix representation of T relative to the standard bases $\mathcal{B} = \{(1, 0)^{\mathrm{T}}, (0, 1)^{\mathrm{T}}\}$ of $\mathbf{R}^2$ and $\mathcal{C} = \{(1, 0, 0)^{\mathrm{T}}, (0, 1, 0)^{\mathrm{T}}, (0, 0, 1)^{\mathrm{T}}\}$ of $\mathbf{R}^3$.

(iv) Use M to find $T(-2, 2)^T$.

(b) (20 points)

(i) Complete the following definition: A set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ in a vector space V is a *basis* for V if ________________

(ii) Does the set $\{(1,2,3)^T, (4,5,6)^T, (2,1,0)^T\}$ form a basis for $\mathbf{R}^3$? Explain.

3. (40 points)

(a) (20 points) Let $\mathbf{y} = (5, -9, -5)^T, \mathbf{u_1} = (-3, -5, 1)^T, \mathbf{u_2} = (-3, 2, 1)^T$.

(i) Show that $\mathbf{u_1}$ and $\mathbf{u_2}$ are orthogonal

(ii) Find the distance from $\mathbf{y}$ to the plane in $\mathbf{R^3}$ spanned by $\mathbf{u_1}$ and $\mathbf{u_2}$.

(b) (20 points) The weather in College Park is either dry (no rain or snow) or wet (some rain or snow). If it is dry one day there is an 80% chance it will be dry the next day. If it is wet one day there is a 40% chance it will be wet the next day.

(i) What is the stochastic matrix for this situation ?

(ii) If Monday is wet, what is the probability that Wednesday will be dry ?

(iii) In the long run, how likely is it that the weather in College Park is dry on a given day ?

4. (30 points) Let

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 7 & 4 \\ 2 & 4 & 7 \end{pmatrix}.$$

The characteristic polynomial of A is $(12 - \lambda)(\lambda - 3)^2$.

(a) Find an orthogonal matrix P such that $P^T AP$ is diagonal.

(b) Classify the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ as positive definite, negative definite, or indefinite.

(c) Find the maximum of $Q(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$.

5. (40 points)

(a) (10 points) Let I be the 85×85 identity matrix.

(i) What is $\det(I)$?

(ii) Let A be the matrix obtained from I by interchanging rows 23 and 67. What is $\det(A)$?

(iii) Let B be the matrix obtained from A by mutiplying rows 9 and 23 by 3. What is $\det(B)$?

(iv) Let C be the matrix obtained from B by adding row 23 to row 11. What is $\det(C)$?

(v) What is $\det(C^T)$?

(b) (15 points) Let $C = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}$. Find numbers $a > 0, b > 0$ and an invertible matrix P such that $P^{-1}CP = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(c) (15 points) Mark each statement as true (T) or false (F).

(i) If none of the vectors in the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subset \mathbf{R^3}$ is a multiple of one of the other vectors, then S is linearly independent.

(ii) If matrices A and B are row equivalent then $\operatorname{rank} A = \operatorname{rank} B$.

(iii) If W is a subspace of R^n then W and $W^\perp$ have no vectors in common

(iv) If A is a 3×3 matrix then $\det(2A) = 2\det(A)$.

(v) If A is a square matrix, A and A^T have the same eigenvalues, counting multiplicities.