Definition of a Matrix

A **matrix** is a rectangular table of numbers.

If m and n are positive integers, then an *m x n* (read “m by n”) matrix is a reactangular array $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

in which every **entry** a_{rc} is a number.

The **order** of the matrix is its size, *m x n*, defined by its number of rows and columns.

A matrix is **Square** if it has the same number of rows and columns, i.e. it is *n x n*.

The **Main Diagonal** of a square matrix includes the entries $a_{11}, a_{22}, \ldots, a_{nn}$.

Matrices and Systems of Linear Equations

A matrix derived from a system of linear equations is called the **Augmented Matrix** of the system.

A matrix derived from the coefficients of the system, not including the constant terms, is called the **Coefficient Matrix** of the system.

Example.

$$System: \begin{cases} x + y - 2z = 1 \\ \quad\;\; y + 3z = 2 \\ \qquad\quad\; z = 0 \end{cases}$$

Note: Systems are written in **Standard Form** w/ constant term on the right.

$$AugmentedMatrix: \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$CoefficientMatrix: \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-Echelon Form of a Matrix

For a square matrix, the matrix is in Row-Echelon Form if there are zeros in its lower triangle and ones on its main diagonal.

Example.

$$RowEchelonForm: \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

A matrix in row-echelon form is in **Reduced Row-Echelon Form** if every column that has a leading one has zeros everywhere else.

$$\mathrm{Re}\,ducedRowEchelonForm: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Systems of Linear Equations with Matrices

1. Gaussian Elimination with Back-substitution

- Convert System of Equations to corresponding Augmented Matrix
- Use Elementary Row Operations (page 546 in text) to find equivalent systems
- Stop when the Coefficient Matrix is in Row-Echelon Form. Now, we know the value of the "last" variable.
- Use Back-substitution to find the values of the other variables.

Example:
$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$z = 0$$
$$y + 3(0) = 2 \rightarrow y = 2$$
$$x + 1(2) - 2(0) = 1 \rightarrow x = -1$$
$$(-1,2,0) = Solution$$

2. Gauss-Jordan Elimination

- Similar to Gaussian Elimination with Back-substitution, however, we continue applying elementary row operations until the Coeffient Matrix is in Reduced Row-Echelon Form.
- Now, we can simply read off the solution.

Example:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$z = 0$$
$$y = 2$$
$$x = -1$$
$$(-1,2,0) = Solution$$

3. Matrix Inversion

- $AX = B \rightarrow A^{-1}AX = A^{-1}B \rightarrow X = A^{-1}B$

The Identity Matrix

The identity matrix of order *n x n* is the square matrix with ones on its main diagonal and zeros everywhere else.

$$I_{nxn} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Operations with Matrices

1. **Matrix Addition** – We can add two matrices of the same order by adding their corresponding entries

$$A + B = [a_{ij}] + [b_{ij}]$$

$$\begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2+1 & (-1)+(-2) \\ 4+(-1) & 0+3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 3 \end{bmatrix}$$

2. **Scalar Multiplication** – We can multiple of matrix by a number (scalar) by multiplying each entry in the matrix by the number.

$$nA = [n \cdot a_{ij}]$$

$$-3 \cdot \begin{bmatrix} 1 & 0 \\ -1 & -2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -3 \cdot 1 & -3 \cdot 0 \\ -3 \cdot -1 & -3 \cdot -2 \\ -3 \cdot 3 & -3 \cdot 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 3 & 6 \\ -9 & -6 \end{bmatrix}$$

3. **Matrix Multiplication** – We can multiply a matrix of order *m x n* times a matrix of order *n x p*. Notice the only requirement is that the number of columns of the first factor is equal to the number of rows in the second factor. The product becomes a matrix of order *m x p* according the following formula:

$A_{mxn} B_{nxp} = C_{mxp}$ where each entry in the product C is determined by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$.

Example. $$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (1)(1)+(2)(4) & (1)(2)+(2)(5) & (1)(3)+(2)(6) \\ (0)(1)+(1)(4) & (0)(2)+(1)(5) & (0)(3)+(1)(6) \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \end{bmatrix}$$

- Note that matrix multiplication is in general NOT commutative, $AB \neq BA$.
- For matrices of the appropriate order, $IA = A$ and $AI = A$.

4. **Matrix Inverse** – The inverse of a square matrix, A, is the matrix, B, such that $AB = I$. If the matrix B exists, then we say that $B = A^{-1}$ (read "A inverse").

- We find the inverse of a square matrix, A, by performing Gauss-Jordan Elimination on A augmented with the identity matrix. When we are finished with G-J, we obtain A^{-1}.
- $[A \vdots I] \rightarrow$ Gauss-Jordan Elimination $\rightarrow [I \vdots A^{-1}]$

Some Properties of Matrix Arithmetic

1. $A + B = B + A$
2. $(A + B) + C = (A + B) + C$
3. $n(A + B) = nA + nB$
4. $A - B = A + (-1)B$
5. $A(BC) = (AB)C$
6. $A(B + C) = AB + AC$
7. $AA^{-1} = I = A^{-1}A$

Determinants

Def: The determinant of a square matrix is a function that takes the matrix as input and gives a real number as output.

- Notation
  - "The determinant of the matrix A":
    - $\det A$
    - $|A|$

- For 2x2 matrices
  - Given a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad - bc$

- For 3x3 and higher order matrices, there are other more complicated formulas.

Determinants and Inverses

- A square matrix **HAS** an inverse if and only if the value of its determinant is not zero.

- Therefore, given a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the matrix A^{-1} exists only when $\det A = ad - bc \neq 0$.

- In this case, we can find $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- Example:

$$B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$\det B = (1)(-1) - (0)(2) = -1$$

$$B^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

This is a rare example when a matrix is equal to its own inverse. However, we can check that we have computed the inverse matrix correctly by verifying that $BB^{-1} = I$ or $B^{-1}B = I$.