Math 241H - Review of Special Integrals

First, we have important choices of coordinates for $\mathbf{R}^2$ and $\mathbf{R}^3$. When you are integrating over circles in $\mathbf{R}^2$, it is usually easiest to use polar coordinates, given by

$$x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta, r \geq 0, 0 \leq \theta \leq 2\pi.$$

In $\mathbf{R}^3$ when you are integrating over solids given in terms of cylinders and paraboloids, it is usually easiest to use cylindrical coordinates given by

$$x = r\cos\theta, y = r\sin\theta, z = z, dV = r\ dzdrd\theta, r \geq 0, 0 \leq \theta \leq 2\pi.$$

In $\mathbf{R}^3$ when you are integrating over solids given in terms of cones and spheres, it is usually easiest to use spherical coordinates given for $\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ by

$$x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi, dV = \rho^2\sin\phi d\rho d\phi d\theta.$$

Next, we have various integrals over curves. The formulas are given for a curve in $\mathbf{R}^3$, but also work for curves in $\mathbf{R}^2$ if you omit the z component. Let C be the curve parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t)),\quad a \leq t \leq b$. Then the length of the curve is given by

$$L = \int_a^b \|\mathbf{r}'(t)\|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt.$$

If $f : \mathbf{R}^3 \to \mathbf{R}$ is a real-valued function $f(x, y, z)$, the scalar line integral of f along the curve C parameterized by $\mathbf{r}$ above is given by

$$\int_C f\ ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt = \int_a^b f(x(t), y(t), z(t))\ \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt.$$

We also have vector line integrals known as work integrals. For these the curve C is oriented. If the orientation is reversed, the integral changes sign. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field in $\mathbf{R}^3$ and C is a curve in $\mathbf{R}^3$ parametrized by $\mathbf{r}$ as above. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Mdx + Ndy + Pdz = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$$

$$= \int_a^b (M(x(t), y(t), z(t))x'(t) + N(x(t), y(t), z(t))y'(t) + P(x(t), y(t), z(t))z'(t))\ dt$$

We have the following shortcut formulas for work integrals. First, if $\mathbf{F}$ is conservative, so that $\mathbf{F} = grad\ f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ is the gradient of a function $f(x, y, z)$, then

$$\int_C Mdx + Ndy + Pdz = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x(b), y(b), z(b)) - f(x(a), y(z), z(a)).$$

Second, if C is a closed curve in the xy plane, oriented counterclockwise, which bounds a region R and $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, then

$$\int_C M dx + N dy = \int\int_R (N_x - M_y)\ dxdy.$$

Finally, if C is a closed curve in $\mathbf{R}^3$ which bounds a surface Σ, we have Stoke's Theorem which is given below in the section on surface integrals.

Next we have various types of integrals over surfaces. Let Σ be a surface parametrized by $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ where (u,v) are in the region R. We will also give formulas for the special case that Σ is the part of the graph of $z = f(x,y)$ where (x,y) are in the region R in the xy-plane.

The surface area of Σ is given by

$$\int\int_R \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\ dudv \quad \text{or} \quad \int\int_R \sqrt{f_x^2 + f_y^2 + 1}\ dxdy.$$

Let $g : \mathbf{R}^3 \to \mathbf{R}$ be a function $g(x,y,z)$. Then the scalar surface integral of g over the surface Σ is given by

$$\int\int_\Sigma g\ dS = \int\int_R g((x(u,v), y(u,v), z(u,v)))\ \|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)\|\ dudv$$

$$\text{or} \quad \int\int_\Sigma g\ dS = \int\int_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\ dxdy.$$

Finally, we have vector surface integrals, also known as flux integrals. For these integrals we are given a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and the surface is oriented. If the orientation is reversed, the integral changes sign. For these we need a normal vector to the surface. In the case of parameterized surfaces, the normal vector is given by $\mathbf{n}(u,v) = \mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)$. For the case of a graph $z = f(x,y)$, the upward pointing normal vector is given by $\mathbf{n}(x,y) = -f_x(x,y)\mathbf{i} - f_y(x,y)\mathbf{j} + \mathbf{k}$. The flux integral is

$$\int\int_\Sigma \mathbf{F} \cdot \mathbf{n}\ dS = \int\int_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{n}(u,v)\ dudv \quad \text{or} \int\int_\Sigma \mathbf{F}(x,y,f(x,y)) \cdot \mathbf{n}(x,y) dxdy.$$

Finally, we have Stoke's Theorem and the Divergence Theorem. If Σ is a surface and C is its complete boundary, oriented consistently, then Stoke's Theorem given an equality between the flux integral of *curl* $\mathbf{F}$ over Σ, and the work integral of $\mathbf{F}$ over the boundary curve C.

$$\int\int_\Sigma \mathit{curl}\ \mathbf{F} \cdot \mathbf{n}\ dS = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Let D be a solid region in $\mathbf{R}^3$ and let Σ be its complete boundary, with outward pointing normal vectors. The Divergence Theorem gives an equality between the flux integral of $\mathbf{F}$ over Σ and the ordinary triple integral of *div* $\mathbf{F} = M_x + N_y + P_z$, the divergence of $\mathbf{F}$, over D.

$$\int\int_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_D \mathit{div}\ \mathbf{F}\ dV = \int\int\int_D (M_x(x,y,z) + N_y(x,y,z) + P_z(x,y,z)) dxdydz.$$