Representation Theory and Harmonic Analysis on Semisimple Lie Groups
Subtitle: Basic Harish-Chandra Theory

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When I began to study representation theory in the early 70's, learning the theory of unitary representaitons of semisimple Lie groups primarily mean learning Harish-Chandra's work. In the past 25 years the field has grown enormously, in many directions. However, Harish-Chandra's work still serves as a foundation for the subject, and I think it is important for students entering the field to at least have an overview of his accomplishments. These notes give an introduction to the representation theory of semisimple Lie groups from the point of view of harmonic analysis, with the goal of explaining Harish-Chandra's Plancherel formula. I will cover the following topics.

1. Basic definitions and motivation from compact and abelian groups.
2. Characters of unitary representations
3. Discrete series representations
4. Parabolic induction
5. Parameterization of tempered representations
6. Plancherel formula and wave packets
7. More about characters

§1. Basic definitions and motivation from compact and abelian groups.

The theory of harmonic analysis on groups originated in the eighteenth century with the problem of representing an arbitrary periodic function by a trigonometric series. In its modern version, the theory of Fourier series can be formulated as follows. Let $\mathbf{R}$ denote the additive group of real numbers, $2\pi\mathbf{Z}$ the subgroup of $\mathbf{R}$ consisting of all integer multiples of 2π, and let $\mathbf{T} = \mathbf{R}/\mathbf{2\pi Z}$ be the quotient group. Then functions on $\mathbf{T}$ can be thought of as functions on the interval $[0, 2\pi]$ or as 2π-periodic functions on the real line. Lebesgue measure on $[0, 2\pi]$ gives a measure on $\mathbf{T}$ which we normalize to have total mass one. Thus we define

$$\int_{\mathbf{T}} f(x)\ dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx.$$

Instead of expanding real-valued functions on $\mathbf{T}$ in terms of the functions $\sin nx, \cos nx, n = 0, 1, 2, ...$, it is more convenient to expand complex-valued functions on $\mathbf{T}$ in terms of the complex exponential functions $e^{inx} = \cos nx + i \sin nx, n \in \mathbf{Z}$.

Suppose $f : \mathbf{T} \mapsto \mathbf{C}$ is an integrable function, that is $f \in L^1(\mathbf{T})$. Then for any integer n we can define a Fourier coefficient

$$\hat{f}(n) = \int_{\mathbf{T}} f(x)\ e^{-inx} dx.$$

We then attach to f the Fourier series

$$f \leftrightarrow \sum_{n=-\infty}^{\infty} \hat{f}(n)\ e^{inx}.$$

The question is, in what sense does the Fourier series represent the function f? One nice formulation is as follows.

Let $L^2(\mathbf{T})$ denote the complex vector space of all measurable functions f such that

$$\|f\|_2 = (\int_{\mathbf{T}} |f(x)|^2 dx)^{\frac{1}{2}} < \infty.$$

We define an inner product on $L^2(\mathbf{T})$ by

$$< f, g >= \int_{\mathbf{T}} f(x)\overline{g(x)} dx.$$

Then $L^2(\mathbf{T})$ is a Hilbert space, and if we set $e_n(x) = e^{inx}$, then $\{e_n\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$. Further, the Fourier coefficient $\hat{f}(n) =< f, e_n >$, so that the Fourier series of $f \in L^2(\mathbf{T})$ is just the expansion of f in terms of this orthonormal basis. Thus the partial sums of the Fourier series converge to f in the L^2 norm, and

$$f = \sum_{n \in \mathbf{Z}} \hat{f}(n)\ e_n$$

can be taken as an equality of L^2 functions. The functions e_n are an especially nice basis for $L^2(T)$ because they have the following two properties. Let $n \in \mathbf{Z}$. Then
(1) $\mathbf{C}e_n$ is a subspace of $L^2(T)$ which is invariant under translations by elements of T. That is, for any fixed $y \in T$,

$$e_n(x+y) = e^{in(x+y)} = e^{iny}e^{inx} = e^{iny}\ e_n(x), x \in T.$$

(2) e_n is the solution of a simple differential equation, $d/dx\ e_n = in\ e_n, n \in \mathbf{Z}$. Thus we can think of the Fourier series as an eigenfunction expansion of f.

I would now like to turn from the theory of harmonic analysis to that of group representations. Assume that G is a locally compact topological group. That is G is a

Hausdorff, locally compact topological space with a group structure which is compatible with the topology in the sense that the group operations are continuous. Let W be a Hilbert space, that is a finite or infinite-dimensional complex vector space with hermitian inner product $<,>$, which is complete with respect to the norm $\|w\| =< w, w >^{\frac{1}{2}}, w \in W$, coming from the inner product. An invertible linear operator $T : W \to W$ is called unitary if $< Tv, Tw >=< v, w >$ for all $v, w \in W$. The unitary operators on W with the operation of composition form a group which we will call $U(W)$. Let $\pi : G \to U(W)$ be a group homomorphism which is continuous in the sense that for every $v, w \in W$, the complex-valued function $x \mapsto < \pi(x)v, w >, x \in G$, is continuous. Then π or (π, W) is called a unitary representation of G.

Suppose that $\dim W = 1$. In this case $U(W)$ is isomorphic to the circle group $S^1 = \{z \in \mathbf{C} : |z| = 1\}$, and using this identification, $\pi : G \to S^1$ can be thought of as a continuous group homomorphism from G to S^1. One-dimensional unitary representations are called unitary characters.

Since G is a locally compact topological group, there is a unique (up to constant factor) right Haar measure on G. The main property of Haar measure is that it is translation invariant. That is, for every $f \in L^1(G), g \in G$,

$$\int_G f(x)\ dx = \int_G f(xg)\ dx = \int_G [R(g)f](x)\ dx$$

where

$$[R(g)f](x) = f(xg), x, g \in G.$$

Let $L^2(G)$ be the Hilbert space of square-integrable functions on G with respect to this measure. Since we use right Haar measure to define the inner product in $L^2(G)$, $R(g)f \in L^2(G)$ for all $g \in G, f \in L^2(G)$, and in fact $R(g)$ is a unitary operator on $L^2(G)$. Now $R : G \to U(L^2(G))$ is a continuous group homomorphism, hence a unitary representation of G. It is called the right regular representation of G.

Let (π, W) be a unitary representation of G. A closed subspace $V \subset W$ is called invariant if $\pi(g)v \in V$ for all $g \in G, v \in V$. In this case we obtain a unitary representation of G on V by restricting the operators $\pi(g), g \in G$, to V. It is called a subrepresentation of π. If V is invariant, so is

$$V^{\perp} = \{w \in W :< v, w >= 0\ \forall v \in V\}.$$

Then $W = V \oplus V^{\perp}$, and we can regard (π, W) as the direct sum of representations of G on V and $V^{\perp}$. Now (π, W) is called irreducible if it has no proper invariant subspaces. For example, if W is one-dimensional, (π, W) is irreducible since W has no proper subspaces. If G is abelian, these are the only irreducible representations. However nonabelian groups have higher dimensional, and even infinite-dimensional irreducible representations.

Suppose that (π_1, W_1) and (π_2, W_2) are two unitary representations of G. We say that (π_1, W_1) and (π_2, W_2) are (unitarily) equivalent if there is an invertible linear operator $T : W_1 \to W_2$ such that $< Tv, Tw >=< v, w >$ for all $v, w \in W_1$ and

$$\pi_2(g) = T\pi_1(g)T^{-1} \ \forall \ g \in G.$$

We regard equivalent representations as being the same.

Two of the most important problems in representation theory are the following.

(1) Given a locally compact group G, find $\hat{G}$, the set of equivalence classes of irreducible unitary representations.

(2) Given an arbitary unitary representation of G, describe how to decompose it into irreducible constituents. The most important case here is the decomposition of the regular representation of G on $L^2(G)$. This is L^2 harmonic analysis on G.

Let $G = \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Then $\mathbf{T}$ is an abelian group, so $\hat{\mathbf{T}}$ is the set of unitary characters of $\mathbf{T}$. For each $n \in \mathbf{Z}$, the function $e_n(x) = e^{inx}, x \in \mathbf{T}$, is a unitary character of $\mathbf{T}$, and these are the only continuous homomorphisms of $\mathbf{T}$ into S^1. Thus $\hat{\mathbf{T}} \simeq \mathbf{Z}$. For each $n \in \mathbf{Z}, \mathbf{C}e_n \subset L^2(\mathbf{T})$ is an invariant subspace of $L^2(\mathbf{T})$ and the L^2 theory of Fourier series says that

$$L^2(\mathbf{T}) = \oplus \sum_{\mathbf{n} \in \mathbf{Z}} \mathbf{Ce_n}$$

is the decomposition of $L^2(\mathbf{T})$ into irreducible subspaces. Further, for each $n \in \mathbf{Z}$, the projection of $L^2(\mathbf{T})$ onto $\mathbf{C}e_n$ is given by $f \mapsto \hat{f}(n)e_n, f \in L^2(\mathbf{T})$.

Suppose that G is a compact group. Then every $(\pi, W) \in \hat{G}$ is finite-dimensional. The dimension d_π of W is called the degree of π. We define a matrix coefficient of $(\pi, W) \in \hat{G}$ to be any function of the form $\phi_{v,w}(x) =< \pi(x)v, w >, v, w \in W$. These are all continuous functions by our continuity requirement on π, and G is compact, so every matrix coefficient is in $L^2(G)$. Let $L^2(G)_\pi$ denote the subspace of $L^2(G)$ spanned by the matrix coefficients of $\pi \in \hat{G}$. It is an invariant subspace of dimension d_π^2 and the restriction of the right regular representation of G to $L^2(G)_\pi$ can be decomposed as the direct sum of d_π irreducible subrepresentations, each of which is equivalent to π. Functions in $L^2(G)_\pi$ are regarded as elementary functions because they transform in the simplest possible way with respect to translations by elements of the group. The Peter-Weyl theorem says that

$$L^2(G) = \oplus \sum_{\pi \in \hat{G}} L^2(G)_\pi .$$

That is, $L^2(G)$ decomposes as the direct sum of irreducible subspaces, and each $\pi \in \hat{G}$ occurs with multiplicity d_π. The inner product on $L^2(G)$ is also determined by the representation theory. If $\pi_1, \pi_2 \in \hat{G}$ and $f_i \in L^2(G)_{\pi_i}, i = 1, 2$, then $< f_1, f_2 >= 0$ unless $\pi_1 \simeq \pi_2$. Suppose $(\pi, W) \in \hat{G}$, and $v_i, w_i \in W, i = 1, 2$. Then

$$< \phi_{v_1,w_1}, \phi_{v_2,w_2} >= d_\pi^{-1} < v_1, v_2 > \overline{< w_1, w_2 >}.$$

The analogue of the Fourier series for a function $f \in L^2(G)$ can now be defined as follows. Let (π, W) be an irreducible unitary representation of G. Then the character of π is the element of $L^2(G)_\pi$ defined by

$$\Theta_\pi(x) = \text{trace } \pi(x) = \sum_i < \pi(x)e_i, e_i >, x \in G,$$

where $\{e_i\}$ denotes an orthonormal basis for W. When $\dim W > 1$, it is not a group homomorphism, but it is a class function on G. The character of π determines π in the sense that irreducible unitary representations π_1 and π_2 are equivalent if and only if their characters are equal.

The projection of $L^2(G)$ onto the subspace $L^2(G)_\pi$ is given by $f \mapsto f_\pi$ where

$$f_\pi(x) = d_\pi(\Theta_\pi * f)(x) = d_\pi \int_G \Theta_\pi(xy^{-1})f(y)dy.$$

If $d_\pi = 1$ so that $\Theta_\pi(x) = \pi(x)$ is a unitary character, then

$$(\pi * f)(x) = \int_G \pi(xy^{-1})f(y)\ dy = \pi(x) \int_G \overline{\pi(y)} f(y)\ dy = \hat{f}(\pi)\pi(x)$$

just as in the case of $G = \mathbf{T}$. Thus we have an expansion of $f \in L^2(G)$ in terms of elementary functions,

$$f = \sum_{\pi \in \hat{G}} d_\pi(\Theta_\pi * f), \quad f \in L^2(G).$$

The equality can be interpreted in the L^2 sense that the partial sums of the infinite series converge to f in $L^2(G)$ or can be taken literally if f is nice enough that the series converges pointwise to f. It is called the Plancherel formula for G and the measure on $\hat{G}$ that assigns to each $\pi \in \hat{G}$ the mass d_π is called the Plancherel measure on $\hat{G}$. Thus the theory of characters of irreducible unitary representations of G yields a theory of L^2 harmonic analysis on G.

Suppose that G is a compact Lie group so that it is possible to differentiate on G as well as integrate. Let $D(G)$ denote the algebra of differential operators on G that commute with both left and right translations. (For example, when $G = T = \mathbf{R}/2\pi\mathbf{Z}$, then $D(T)$ consists of polynomials in d/dx.) Then for each $\pi \in \hat{G}$ there is a homomorphism $\chi_\pi : D(G) \to \mathbf{C}$ so that for ϕ any matrix coefficient of π, $D\ \phi = \chi_\pi(D)\ \phi$ for all $D \in D(G)$. That is the above decomposition

$$f = \sum_{\pi \in \hat{G}} d_\pi(\Theta_\pi * f), \quad f \in L^2(G),$$

can also be regarded as an eigenfunction expansion for $D(G)$.

Let G be any locally compact abelian group. Then all its irreducible unitary representations are one-dimensional, that is are unitary characters. However, if G is non-compact, these characters are not L^2 functions, since they have constant absolute value one. In particular, in this case $L^2(G)$ has no irreducible subrepresentations. But $L^2(G)$ can still be decomposed in terms of these characters as a “direct integral” rather than as a direct sum. This generalizes the theory of the Fourier transform for functions on the real line. For $f \in L^1(G) \cap L^2(G)$ we can define Fourier coefficients

$$\hat{f}(\pi) = \int_G f(x)\overline{\pi(x)}\ dx, \quad \pi \in \hat{G}.$$

Further, the set $\hat{G}$ of unitary characters is itself a locally compact abelian group so it has a Haar measure. The Plancherel theorem says that the Haar measures dx on G and $d\pi$ on $\hat{G}$ can be normalized so that $f \mapsto \hat{f}$ extends to an isometry of $L^2(G)$ onto $L^2(\hat{G})$. That is, for every $f \in L^2(G)$, $\hat{f} \in L^2(\hat{G})$ and

$$\int_G |f(x)|^2 dx = \int_{\hat{G}} |\hat{f}(\pi)|^2 d\pi.$$

Just as before there is a Fourier inversion formula which says that if $f \in L^2(G)$ is sufficiently nice,

$$f(x) = \int_{\hat{G}} \hat{f}(\pi)\pi(x) d\pi.$$

Thus we can think of

$$L^2(G) = \oplus \int_{\hat{G}} \mathbf{C}\pi \; d\pi.$$

Of course a one-dimensional representation is its own character, and it is easy to check that

$$\hat{f}(\pi)\pi(x) = (\pi * f)(x), x \in G,$$

where the convolution is defined as in the compact case. The Plancherel formula is

$$f = \int_{\hat{G}} (\pi * f) d\pi$$

as in the compact case. Thus the Plancherel measure on $\hat{G}$ is just the Haar measure and we can think of $L^2(G)$ as a direct integral of irreducible representations with respect to Haar measure.

For example, suppose that $G = \mathbf{R}$ is the additive group of real numbers. For each $y \in \mathbf{R}, \phi_y(x) = e^{ixy}, x \in \mathbf{R}$, is a unitary character of $\mathbf{R}$, and all unitary characters are of this form. Thus $\hat{G} \simeq \mathbf{R} \simeq G$ so that G is self-dual. Further,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(t)|^2 \; dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \; dt, f \in L^2(\mathbf{R}),$$

where dt denotes Lebesgue measure on $\mathbf{R}$. That is we can take $dx = d\pi = (2\pi)^{-1/2} \; dt$ in this case. Since G is non-compact, none of its unitary characters are L^2 functions. Thus $L^2(\mathbf{R})$ has no irreducible invariant subspaces. However $L^2(\mathbf{R})$ has plenty of invariant subspaces. Let C be a closed subset of $\mathbf{R}$ and define $V(C)$ to be the set of all $f \in L^2(\mathbf{R})$ such that the support of $\hat{f}$ is contained in C. Then $V(C)$ is a closed invariant subspace of $L^2(\mathbf{R})$. For $y \in \mathbf{R}$, if the character ϕ_y were an L^2 function, $\hat{\phi}_y$ would have to be given by $\hat{\phi}_y(y) = 1, \hat{\phi}_y(x) = 0, x \neq y$, and $\mathbf{C}\phi_y$ would be equal to $V(\{y\})$. Of course, this does not make sense since an L^2 function supported at one point is the zero function as an element of L^2. However, we can find L^2 functions whose Fourier transforms are supported in a small neighborhood of y. Take $\alpha \in C_c^\infty(\mathbf{R})$ with support in a small compact set C containing a neighborhood of y. Then

$$f_\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha(t) \; e^{ixt} \; dt$$

is an element of $V(C)$. It is called a wave packet.

§2. Characters of unitary representations.

We now turn to the class of semisimple Lie groups. Although there is a more general definition of Lie group, for simplicity we will only consider linear Lie groups. For $n \geq 1$, let $GL(n, \mathbf{R})$ and $GL(n, \mathbf{C})$ denote the groups of invertible $n \times n$ matrices with real or complex entries. They have natural topologies as open subsets of $\mathbf{R}^{n^2}$ or $\mathbf{C}^{n^2}$. Any closed subgroup of $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$ is called a linear Lie group. Let G be a connected, linear Lie group. Then G is called reductive if it is stable under conjugate transpose. Further, G is called semisimple if it is reductive, nonabelian, and has finite center. For example, $GL(n, \mathbf{C})$ is reductive, but not semisimple, since its center, the set of all scalar matrices with nonzero determinant, is isomorphic to $C^{\times}$. However $G = SL(n, \mathbf{C}) = \{g \in GL(n, \mathbf{C}) : \det g = 1\}$ is semisimple for $n \geq 2$ because the condition $\det g = 1$ is stable under conjugate transpose, and the center of G, the set of all scalar matrices with determinant one, is isomorphic to the finite group of complex n^{th} roots of unity. $SL(1, \mathbf{C}) = \{1\}$ is not semisimple since it is abelian.

The class of connected linear reductive Lie groups contains all connected Lie groups which are either compact or abelian. It also contains the classical families of complex and real semisimple Lie groups such as the various orthogonal, symplectic, and unitary groups, for example $SL(n, \mathbf{C}), SO(n, \mathbf{C}), Sp(2n, \mathbf{C}), SL(n, \mathbf{R}), SU(m, n), Sp(2n, \mathbf{R})$.

Let G be a connected linear semisimple Lie group. Two basic problems of harmonic analysis are to find $\hat{G}$, the set of equivalence classes of irreducible unitary representations, and then to find the Plancherel measure on $\hat{G}$ that gives the decomposition of the regular representation of G on $L^2(G)$ as a direct sum or integral of irreducible constituents. The first of these problems is still not completely solved. However the second was solved by Harish-Chandra in a series of papers dating from the early 50's to the mid 70's. You might think that it would be necessary to find $\hat{G}$ before finding the Plancherel measure. However, what Harish-Chandra did was find enough of $\hat{G}$ to prove the Plancherel theorem; that is he found all of $\hat{G}$ except a set of Plancherel measure zero.

Recall that for compact and abelian groups we can write the Plancherel formula as

$$f = \int_{\hat{G}} (\Theta_\pi * f) \, d\pi, \; f \in L^2(G),$$

where $\Theta_\pi(x) = \text{trace } \pi(x), x \in G$, is the character of $\pi \in \hat{G}$ and $d\pi$ is Plancherel measure on $\hat{G}$. It was possible to define the character in these cases because all irreducible unitary representations were finite-dimensional. However, for a noncompact simple Lie group, the only finite-dimensional irreducible unitary representation is the trivial unitary character

$\pi(x) = 1, x \in G$. Thus the first problem in generalizing the theory for compact and abelian groups to semisimple Lie groups is to define the character of an infinite-dimensional irreducible unitary representation $\pi \in \hat{G}$. In this case the unitary operators $\pi(x), x \in G$, do not have a well-defined trace since they have infinitely many eigenvalues, all of absolute value one.

Let (π, W) be an irreducible unitary representation of G and let $f \in C_c^\infty(G)$, the set of smooth compactly supported functions on G. For every $v, w \in W$, the matrix coefficient $< \pi(x)v, w >$ is a continuous function on G, so that the product $f(x) < \pi(x)v, w >$ is continuous and compactly supported, hence integrable on G. We define an operator $\pi(f)$ on W by

$$< \pi(f)v, w > = \int_G f(x) < \pi(x)v, w > dx, \;\; v, w \in W.$$

It can also be written with an operator-valued integral as

$$\pi(f) = \int_G f(x)\pi(x) \; dx.$$

The operator $\pi(f)$ is not unitary, but is of trace class. That is

$$\Theta_\pi(f) = \text{trace } \pi(f) = \sum_{e_i} < \pi(f)e_i, e_i >$$

converges and is independent of the orthonormal basis $\{e_i\}$ of W. Further, $f \mapsto \Theta_\pi(f)$ is a distribution on G, that is a continuous linear functional on $C_c^\infty(G)$. This distribution is called the character of π. As in the compact case, if π_1 and π_2 are any irreducible unitary representations of G, $\Theta_{\pi_1} = \Theta_{\pi_2}$ if and only if π_1 and π_2 are equivalent. Thus if $\pi \in \hat{G}$, we can write Θ_π for the character of any representative of the equivalence class π. We say that $\pi \in \hat{G}$ is tempered if the distribution Θ_π extends continuously to $\mathcal{C}(G)$, the Schwartz space of functions $f \in C^\infty(G)$ such that f and all its derivatives are L^2 functions on G.

Suppose that W is finite-dimensional. Then we have two definitions of the character Θ_π of π, one as a distribution, $\Theta_\pi(f) = \text{trace}\pi(f), f \in C_c^\infty(G)$, and one as a function $\Theta_\pi(x) = \text{trace } \pi(x), x \in G$. The relation between the two is that

$$\Theta_\pi(f) = \int_G \Theta_\pi(x)f(x) \; dx, \quad f \in C_c^\infty(G).$$

Although the characters of infinite-dimensional unitary representations must be defined initially as distributions, they are also given by integration against locally summable functions. That is, given $\pi \in \hat{G}$ there is a function Θ_π on G which is integrable over any compact subset of G and which satisfies

$$\Theta_\pi(f) = \int_G f(x)\Theta_\pi(x)\ dx, \quad f \in C_c^\infty(G).$$

Further, there is a dense open subset G' of G called the set of regular elements of G such that Θ_π is analytic on G' for all $\pi \in \hat{G}$. This is known as the regularity theorem.

Because of the regularity theorem, for $f \in C_c^\infty(G)$ we can define

$$(\Theta_\pi * f)(x) = \int_G \Theta_\pi(xy^{-1})\ f(y)\ dy,\ x \in G,$$

just as in the compact case. Then $\Theta_\pi * f$ is in the span of the matrix coefficients of π and is an eigenfunction for $D(G)$. We say that μ is the Plancherel measure on $\hat{G}$ if

$$f(x) = \int_{\hat{G}} (\Theta_\pi * f)(x)\ d\mu(\pi), \quad x \in G, f \in C_c^\infty(G).$$

To make the Plancherel theorem precise, we need to describe $\hat{G}$ and the measure $d\mu$. In order to do this we need two ingredients, discrete series representations and parabolic induction.

§3. Discrete series representations.

Let G be a semisimple Lie group. A representation $\pi \in \hat{G}$ is called a discrete series representation of G if it has a non-zero matrix coefficient which is square-integrable. In this case every matrix coefficient of π is square-integrable, and as in the compact case we let $L^2(G)_\pi$ denote the subspace of $L^2(G)$ spanned by the matrix coefficients of π. Write $\hat{G}_d \subset \hat{G}$ for the set of equivalence classes of discrete series representations of G. Then one piece of $L^2(G)$ is given by

$$L^2_d(G) = \sum_{\pi \in \hat{G}_d} L^2(G)_\pi.$$

As for compact groups, for $\pi_1, \pi_2 \in \hat{G}_d$, $L^2(G)_{\pi_1}$ is orthogonal to $L^2(G)_{\pi_2}$ unless $\pi_1 \simeq \pi_2$. Further, for $(\pi, W) \in \hat{G}_d$ there is a number d_π called the formal degree of π (recall that W is infinite-dimensional, so π does not have a degree) so that for all $v_1, w_1, v_2, w_2 \in W$, just as for compact groups,

$$< \phi_{v_1,w_1}, \phi_{v_2,w_2} >= d_\pi^{-1} < v_1, v_2 > \overline{< w_1, w_2 >}.$$

When G is non-compact, $L^2(G)_d$ is not all of $L^2(G)$. It may even be $\{0\}$.

The motivation for the parameterization of discrete series representations of noncompact semisimple Lie groups comes from the theory of compact Lie groups. Suppose that G is a compact connected Lie group. An abelian subgroup B of G is called a Cartan subgroup (or maximal torus) of G if no larger abelian subgroup of G properly contains it. The Cartan subgroups of G are compact connected abelian Lie groups, and are all conjugate inside G. Fix a Cartan subgroup B of G. Then the set $\hat{B}$ of unitary characters of B is a locally compact group with the discrete topology. Let $W(G,B) = N_G(B)/B$, where $N_G(B)$ denotes the normalizer of B in G. Then $W(G,B)$ is a finite group. Every $w \in W(G,B)$ acts on B and hence on $\hat{B}$ by $w\chi(b) = \chi(w^{-1}b), b \in B, \chi \in \hat{B}$. Let $\hat{B}'$ be the set of all $\chi \in \hat{B}$ which are not fixed by any nontrivial $w \in W(G,B)$. Now $\hat{G}$ can be parameterized by the orbits of $W(G,B)$ in $\hat{B}'$. That is, given $\chi \in \hat{B}'$ there is an irreducible unitary representation π_χ of G. Every irreducible unitary representation π of G is equivalent to π_χ for some $\chi \in \hat{B}'$, and for $\chi, \chi' \in \hat{B}'$, π_χ is equivalent to $\pi_{\chi'}$ if and only if there is $w \in W(G,B)$ such that $\chi' = w\chi$.

Now let G be a connected linear semisimple Lie group. An element $g \in G$ is called semisimple if it is semisimple as a matrix, that is it can be diagonalized over the field of complex numbers. A closed subgroup H of G is called a Cartan subgroup if it is a maximal abelian subgroup consisting of semisimple elements. For example, when $G = GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$, the set of all diagonal matrices in G is a Cartan subgroup of G. If G is compact or is a complex Lie group, then all Cartan subgroups of G are connected and are conjugate inside G. However, in the general case, G has finitely many Cartan subgroups up to conjugacy, and Cartan subgroups can have finitely many connected components. Harish-Chandra proved that G has discrete series representations just in case it has a compact Cartan subgroup.

Assume that G has a compact Cartan subgroup B. Then B is unique up to conjugacy, and is a connected compact abelian group. Write $\hat{B}$ for the dual group of unitary characters of B. The discrete series representations of G can be parameterized by a subset $\hat{B}'$ of $\hat{B}$. There is finite group $W_{\mathbf{C}}$ called the complex Weyl group of G which acts on $\hat{B}$, and $\chi \in \hat{B}'$ just in case χ is not fixed by any non-trivial $w \in W_{\mathbf{C}}$. Finally, the real Weyl group $W(G,B) = N_G(B)/B$ is a finite group that acts on $\hat{B}'$, and for $\chi, \chi' \in \hat{B}'$, the discrete series representations of G parameterized by χ and χ' are equivalent just in case $\chi' = w\chi$ for some $w \in W(G,B)$. Note that when G is compact, the complex Weyl group $W_{\mathbf{C}}$ is equal to $W(G,B)$, but in general $W(G,B) \subset W_{\mathbf{C}}$.

Example 1. Let $G = SU(2)$, the compact group of two-by-two unitary matrices. Every Cartan subgroup of G is conjugate

$$B = \{b(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbf{R}\} \simeq \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}.$$

Thus $\hat{B} = \{\chi_n : n \in \mathbf{Z}\}$ where $\chi_n(b(\theta)) = e^{in\theta}$. The Weyl group $W(G, B)$ has one non-trivial element w that maps $b(\theta)$ to $b(-\theta)$. Thus $w\chi_n = \chi_{-n}$ so that $\hat{B}' = \{\chi_n : n \neq 0\}$. Let π_n denote the irreducible representation of G parameterized by $\chi_n, n \neq 0$. Then $\pi_n \simeq \pi_{-n}$, so that $\hat{G} = \{\pi_n : n > 0\}$.

Example 2. Let $G = SL(2, \mathbf{C})$, the group of two-by-two matrices with complex entries and determinant one. Then every Cartan subgroup of G is conjugate to the group D of diagonal matrices

$$D = \{d(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbf{C}^\times\} \simeq \mathbf{C}^\times.$$

Since D is not compact, G has no discrete series representations.

Example 3. Let $G = SL(2, \mathbf{R})$, the group of two-by-two matrices with real entries and determinant one. Then G has compact Cartan subgroup

$$B = \{b(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} : \theta \in \mathbf{R}\} \simeq \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}.$$

Thus $\hat{B} = \{\chi_n : n \in \mathbf{Z}\}$ where $\chi_n(b(\theta)) = e^{in\theta}$. The complex Weyl group $W_{\mathbf{C}}$ has one non-trivial element w that maps $b(\theta)$ to $b(-\theta)$. Thus as in Example 1, $\hat{B}' = \{\chi_n : n \neq 0\}$. However $W(G, B)$ is trivial in this case. Thus the discrete series representations π_n of G parameterized by $\chi_n, n \neq 0$, are all inequivalent.

§4. Parabolic induction.

If H is a subgroup of G, there is a process called induction which takes representations of H and produces representations of G. There is a class of subgroups of G called parabolic subgroups which have especially good properties for induction. The parabolic subgroups P have the form $P = LN$ where L is a reductive Lie group, N is a group of unipotent matrices which is normal in P, and $L \cap N = \{1\}$. If σ is a representation of L, we can extend σ to P by setting $\sigma(xn) = \sigma(x), x \in L, n \in N$, and then induce from P to G. This induction can be normalized so that it takes unitary representations of L to unitary representations of G. Further, if σ is irreducible and tempered, the induced representation of G is tempered, and usually irreducible. In some cases all tempered representations of G

can be obtained in this way from tempered representations of the smaller reductive group L.

Parabolic induction is defined as follows. Let $P = LN$ be a parabolic subgroup of G. Let (σ, V) be any irreducible unitary representation of M. Then we can extend σ to P by $\sigma(xn) = \sigma(x), x \in M, n \in N$. We next induce this representation from P to G as follows.

The groups G, L, and N are all unimodular groups. That is, right Haar measure is also invariant under left translations. However, the parabolic subgroup $P = LN$ is not. If dx is right Haar measure on P so that $d(xp) = dx, p \in P$, then there is a function δ_P on P such that $d(px) = \delta_P(p)dx, p \in P$. It is called the right modular function of P. Let $C(G, V)$ denote the space of all continuous V-valued functions on G, and define

$$C(\sigma) = \{f \in C(G, V) : f(px) = \delta_P^{\frac{1}{2}}(p)\ \sigma(p)\ f(x)\ \forall\ x \in G, p \in P\}.$$

Since the transformation property of $f \in C(\sigma)$ is on the left, the right translates $R(g)f \in C(\sigma)$ for all $g \in G$. Thus G acts on the vector space $C(\sigma)$ by right translations. $C(\sigma)$ is not a Hilbert space. However, there is a maximal compact subgroup K of G such that $G = PK$. Every $f \in C(\sigma)$ is determined by its values on K, and we define a norm on $C(\sigma)$ by

$$\|f\|^2 = \int_K |f(k)|^2 dk,\ \ f \in C(\sigma),$$

where dk is Haar measure on K. Now the Hilbert space $\mathcal{H}(\sigma)$ of the induced representation is the completion of $C(\sigma)$ with respect to this norm. The modular function δ_P is needed in the definition of $C(\sigma)$ in order that the representation of G on $\mathcal{H}(\sigma)$ by right translations be unitary. We will denote this representation by $\mathrm{Ind}_P^G(\sigma)$.

Example 1. Let $G = SU(2)$, the compact group of two-by-two unitary matrices. Then G has no proper parabolic subgroups.

Example 2. Let $G = SL(2, \mathbf{C})$. Then up the conjugacy, the only proper parabolic subgroup of G is $P = LN$ where

$$L = D = \{d(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbf{C}^\times\}, \quad N = \{\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} : w \in \mathbf{C}\}.$$

In this case $L = D$ is abelian and is a noncompact Cartan subgroup of G. For each $n \in \mathbf{Z}, w \in \mathbf{C}$, define

$$\chi(w : n)(d(z)) = (\frac{z}{|z|})^n\ |z|^w, \ \ z \in \mathbf{C}^\times.$$

Then $\chi(w : n)$ is a (not necessarily unitary) character of D. It is unitary just in case $w = i\nu \in i\mathbf{R}$. If we extend these characters trivially to P, and induce up to G we obtain representations $\pi(w : n)$ of G for all $w \in \mathbf{C}^\times, n \in \mathbf{Z}$. When $w = i\nu \in i\mathbf{R}$ so that $\chi(w : n)$ is a unitary character of D, we obtain the family of tempered representations $\pi(i\nu : n), \nu \in \mathbf{R}, n \in \mathbf{Z}$. They are all irreducible, and this gives all irreducible tempered representations of G. The only equivalences among this family of representations are that $\pi(i\nu : n)$ is equivalent to $\pi(-i\nu : -n)$. When $n = 0$ and $0 < w < 2$ is real, the induced representations $\pi(0 : w)$ are also unitary. These representations are nontempered and are called the complementary series for $SL(2, \mathbf{C})$. The only other irreducible unitary representation of G is the trivial representation. These representations do not occur in the Plancherel formula.

Example 3. Let $G = SL(2, \mathbf{R})$. Then $P = LN$ is a proper parabolic subgroup of G where $L = D$ and N are defined as in Example 2 using real numbers in place of complex numbers in the matrices. Again, unitary characters of the abelian group $L = D$ can be extended trivially to P and induced to G to obtain a family of tempered representations of G. These representations are not discrete series, and this family of induced representations together with the discrete series representations gives all tempered representations of G.

§5. Parameterization of tempered representations.

Let G be a connected, semisimple, linear Lie group. In §3 we saw that the discrete series representations of G, if they exist, are parameterized by unitary characters of a compact Cartan subgroup B. In §4 we saw that the tempered induced representations for the groups $SL(2, \mathbf{C})$ and $SL(2, \mathbf{R})$ are also parameterized by the unitary characters of a Cartan subgroup, in this case the noncompact group D of diagonal matrices. In general, every Cartan subgroup H of G will correspond to a series of tempered representations of G, parameterized by the unitary characters of H. When H is compact, these representations will be discrete series. When H is noncompact, these representations will be induced from a proper parabolic subgroup of G.

Let H be a Cartan subgroup of G. Then H is an abelian Lie group with finitely many connected components, and can be decomposed as $H = T \times A$ where T is compact (with finitely many connected components) and $A \simeq \mathbf{R}^n$. Let $L = C_G(A)$, the centralizer of A in G. Then L is a Levi subgroup of G. That is there is a parabolic subgroup $P = LN$ of G (not unique) with L as Levi subgroup. Further, L can be decomposed (not uniquely) as $L = MA$ where M is a reductive Lie group with compact Cartan subgroup T. Although M

need not be semisimple, and can have finitely many connected components, it has discrete series representations which are parameterized just as in §3 by the subset $\hat{T}'$ of $\eta \in \hat{T}$ which are not fixed by any non-trivial element of the complex Weyl group of M. Let σ_η denote the discrete series representation of M parameterized by $\eta \in \hat{T}'$. Then for any $\eta \in \hat{T}', \nu \in \hat{A}$, $\chi(ta) = (\eta \otimes \nu)(ta) = \eta(t)\nu(a), t \in T, a \in A$, is a unitary character of H, and $\sigma_\chi(ma) = \sigma_\eta(m)\nu(a), m \in M, a \in A$, is an irreducible unitary representation of L. Let $\hat{H}'$ denote the set of all $\chi \in \hat{H}$ such that $\chi = \eta \otimes \nu, \eta \in \hat{T}', \nu \in \hat{A}$. Now the family of tempered representations corresponding to H is the set of induced representations $Ind_P^G(\sigma_\chi), \chi \in \hat{H}'$.

For $\chi, \chi' \in \hat{H}'$, $Ind_P^G(\sigma_\chi)$ is equivalent to $Ind_P^G(\sigma_{\chi'})$ if and only if $\chi' = w\chi$ for some $w \in W(G,H) = N_G(H)/H$. Write $\Theta(H : \chi)$ for the character of $Ind_P^G(\sigma_\chi), \chi \in \hat{H}'$. Although these representations depend on the choice of parabolic subgroup P corresponding to H, the equivalence class of $Ind_P^G(\sigma_\chi)$ and the character $\Theta(H : \chi)$ are independent of P. The family of characters $\Theta(H : \chi), \chi \in \hat{H}'$, depends only on the conjugacy class of H in G. Thus we think of tempered representations as coming in families parameterized by conjugacy classes of Cartan subgroups of G.

§6. The Plancherel formula and wave packets.

Using the notation of §5, the Plancherel formula can be written as

$$f(x) = \sum_H \int_{\hat{H}} (\Theta(H : \chi) * f)(x)\ \mu(H : \chi)\ d\chi, \quad f \in C_c^\infty(G), x \in G.$$

Here H runs over a complete set of representatives for conjugacy classes of Cartan subgroups of G and $d\chi$ is Haar measure on $\hat{H}$. Further, $\mu(H : \chi)$ is a function on $\hat{H}$ which is zero for $\chi \notin \hat{H}'$ and satisfies $\mu(H : w\chi) = \mu(H : \chi)$ for all $\chi \in \hat{H}', w \in W(G,H)$. That is, the Plancherel measure for the series of tempered representations parameterized by $\hat{H}'$ is absolutely continuous with respect to Haar measure on $\hat{H}$. Note that if $H = T \times A$ where T is compact and $A \simeq \mathbf{R}^n$, then $\hat{H}$ has discrete parameters coming from $\hat{T}$ and continuous parameters coming from $\hat{A}$. Thus the Haar measure on $\hat{H}$ is a combination of counting measure for the discrete variables and Lebesgue measure for the continuous variables.

The functions $\mu(H : \chi)$ were determined explicitly by Harish-Chandra. In the case that $H = T$ is compact so that $\hat{H}$ is discrete, the point mass $\mu(H : \chi)$ associated to $\chi \in \hat{H}'$ is the formal degree of the discrete series representation corresponding to χ. In the general situation, when $H = T \times A$ and $\chi \in \hat{H}$ is given by $\chi = \eta \otimes \nu$ were $\eta \in \hat{T}'$ and $\nu \in \hat{A}$, one factor of $\mu(H : \chi)$ is the formal degree of the discrete series representation of M

parameterized by η. The other part is a product of factors similar to the terms occuring in the examples below.

The fact that any $f \in C_c^\infty(G)$ can be recovered from the functions $\Theta(H : \chi) * f$ as H ranges over Cartan subgroups of G and χ ranges over $\hat{H}'$ means that these series of representations are enough for the Plancherel formula. That is the remaining irreducible unitary representations have Plancherel measure zero.

Example 1. Let $G = SU(2)$. Every Cartan subgroup of G is conjugate to the compact Cartan subgroup B of diagonal matrices in G. Using the notation of Example 1 of §3, the only irreducible unitary representations of G (up to equivalence) are the representations $\pi_n, n \geq 1$, parameterized by characters χ_n of the Cartan subgroup B. The representation π_n has degree n so that the Plancherel formula for G is

$$f(x) = \sum_{n=1}^{\infty} n \ (\Theta_{\pi_n} * f)(x), \quad x \in G, f \in C_c^\infty(G).$$

Example 2. Let $G = SL(2, \mathbf{C})$. Every Cartan subgroup of G is conjugate to the group D of diagonal matrices in G. In the the notation of Example 2 of §3, we have the family of tempered induced representations $\pi(i\nu : n), \nu \in \mathbf{R}, n \in \mathbf{Z}$ parameterized by the characters of D. Let $\Theta(D : i\nu : n)$ denote the character of $\pi(i\nu : n)$.

The Plancherel formula for $G = SL(2, \mathbf{C})$ was proven by Gelfand and Naimark in 1950 and says that for every $f \in C_c^\infty(G), x \in G$, we have

$$f(x) = \sum_{n \in \mathbf{Z}} \int_{-\infty}^{\infty} (\Theta(D : i\nu : n) * f)(x) \ (n^2 + \nu^2) \ d\nu$$

where $d\nu$ is Lebesgue measure on $\mathbf{R}$.

Example 3. Let $G = SL(2, \mathbf{R})$. Then every Cartan subgroup of G is conjugate to one of

$$D = \{d(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}^\times\} \simeq \mathbf{R}^\times;$$

$$B = \{b(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbf{R}\} \simeq \mathbf{R}/2\pi\mathbf{Z}.$$

The Cartan subgroup B is compact, and as in §3, Example 3, $\hat{B}' = \{\chi_n : n \in \mathbf{Z}, n \neq 0\}$. Let $\Theta(B : n)$ denote the character of the discrete series representation π_n of G parameterized by $\chi_n \in \hat{B}'$.

The Cartan subgroup $D = T \times A$ where $T = \{\pm I\}$ is compact and $A = \{d(e^t) : t \in \mathbf{R}\}$ is isomorphic to the additive group of real numbers. In this case $L = C_G(A) = D$, $L = MA$ where $M = T$, and we can take $P = LN$ where

$$N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}\}.$$

Further, $\hat{D} = \{\chi(i\nu : \epsilon) : \nu \in \mathbf{R}, \epsilon \in \{0, 1\}\}$ where

$$\chi(i\nu : \epsilon)(d(a)) = (\frac{a}{|a|})^{\epsilon} \ |a|^{i\nu}, \quad a \in \mathbf{R}^{\times}, \nu \in \mathbf{R}, \epsilon \in \{0, 1\}.$$

The complex Weyl group of $M = T$ is trivial, so that $\hat{D}' = \hat{D}$ and the representation of $L = D$ parameterized by $\chi \in \hat{D}$ is just χ itself. Let $\Theta(D : \nu : \epsilon)$ denote the character of the induced representation $\pi(\nu : \epsilon) = Ind_P^G(\chi(i\nu : \epsilon))$. In this case $W(G, D)$ has one non-trivial element w satisfying $w\chi(i\nu : \epsilon) = \chi(-i\nu : \epsilon)$, and so $\pi(\nu : \epsilon)$ is equivalent to $\pi(-\nu : \epsilon)$. Finally, the representations $\pi(\nu : \epsilon)$ are all irreducible except for the representation $\pi(0 : 1)$ which is the sum of two irreducible subrepresentations called limit of discrete series representations.

The Plancherel formula for $G = SL(2, \mathbf{R})$, proven separately by Harish-Chandra in 1952, says that for every $f \in C_c^{\infty}(G), x \in G$, we have

$$f(x) = \sum_{n \in \mathbf{Z}} |n|(\Theta(B : n) * f)(x)$$

$$+\frac{1}{4}\int_{-\infty}^{\infty} (\Theta(D : \nu : 0) * f)(x) \ \nu \ \tanh(\frac{\pi\nu}{2}) \ d\nu + \frac{1}{4}\int_{-\infty}^{\infty} (\Theta(D : \nu : 1) * f)(x) \ \nu \ \coth(\frac{\pi\nu}{2}) \ d\nu.$$

In particular, the formal degree of the discrete series representation π_n is $|n|$.

Recall that the Schwartz space $\mathcal{C}(G)$ is the space of all smooth functions f on G such that f and all its derivatives are L^2 functions on G. As part of the proof of the Plancherel formula, Harish-Chandra constructed Schwartz class functions corresponding to the different series of tempered representations which are the analogues of the wave packets on the real line discussed in §1.

Let $H = T \times A$ be a Cartan subgroup, and let $L = MA = C_G(A)$. Recall that the series of tempered representations corresponding to H has two parameters, a discrete parameter $\eta \in \hat{T}'$, and a continuous parameter $\nu \in \hat{A} \simeq \mathbf{R}^n$. Fix $\eta \in \hat{T}'$ and let σ_η be the discrete series representation of M corresponding to η and $\sigma_{\eta,\nu} = \sigma_\eta \otimes \nu$ be the representation of $L = MA$ corresponding to η and ν. Let $P = LN$ be a parabolic subgroup of G with Levi subgroup $L = MA$ and let K be a maximal compact subgroup of G such that $G = KP$. Then there is a Hilbert space $\mathcal{H}_\eta \subset L^2(K)$ depending only on η and for each $\nu \in \hat{A}$ a representation $\pi_{\eta,\nu}$ of G on $\mathcal{H}_\eta$ such that $\pi_{\eta,\nu}$ is equivalent to the induced representation $Ind_P^G(\sigma_{\eta,\nu})$ for all $\nu \in \hat{A}$. That is, for fixed $\eta \in \hat{T}'$, the induced representations $Ind_P^G(\sigma_{\eta,\nu})$ can be realized on the same Hilbert space $\mathcal{H}_\eta$ for all ν. Of course, now the action of G depends on ν instead of just being right translation, but the action of K is independent of ν. Let $(\mathcal{H}_\eta)^K$ denote the set of all elements of $\mathcal{H}_\eta$ which are contained in a finite-dimensional K-invariant subspace of $\mathcal{H}_\eta$. It is called the set of K-finite vectors in $\mathcal{H}_\eta$ and is a dense subset of $\mathcal{H}_\eta$. Now for any $v, w \in \mathcal{H}_\eta$ we can define a family of matrix coefficients for the representations $\pi_{\eta,\nu}$ by taking

$$\phi_{v,w}(x,\nu) =< \pi_{\chi,\nu}(x)v, w >, x \in G, \nu \in \hat{A}.$$

These matrix coefficients are smooth functions of both x and ν if $v, w \in (\mathcal{H}_\eta)^K$. Now for any $\alpha \in C_c^\infty(\hat{A})$ we can define a wave packet corresponding to $v, w \in (\mathcal{H}_\eta)^K$ by

$$\phi_\alpha(x) = \int_{\hat{A}} \alpha(\nu)\phi_{v,w}(x,\nu)\mu(\eta,\nu)d\nu, x \in G.$$

Harish-Chandra proved that these wave packets are Schwartz class functions on G, and that if we let $\mathcal{C}_H(G)$ denote the subspace of $\mathcal{C}(G)$ spanned by wave packets coming from the Cartan subgroup H, then

$$\mathcal{C}(G) = \oplus \sum_H \mathcal{C}_H(G)$$

where H runs over a complete set of representatives for the conjugacy classes of Cartan subgroups of G. Of course if $H = T$ is compact so that $A = \{1\}$, then there are no continuous parameters, and $\mathcal{C}_H(G)$ is just spanned by the matrix coefficients corresponding to K-finite vectors of the discrete series representations of G.

§6. More on characters.

Harish-Chandra proved that the characters of irreducible unitary representations have the following general properties. Let $\pi \in \hat{G}$, and let Θ_π denote its character as a function on the regular set G' of G. Then, just as in the compact case,

$$\Theta_\pi(yxy^{-1}) = \Theta_\pi(x), y \in G, x \in G'.$$

Thus Θ_π is constant on conjugacy classes in G', and so is determined by its restriction to a set of representatives for the conjugacy classes in G'. These can be described as follows.

First, G' is the set of all $g \in G$ such that g is contained in exactly one Cartan subgroup H of G. The fact that $g \in G'$ is contained in a Cartan subgroup implies that g is semisimple. The fact that it contained in only one Cartan subgroup implies the centralizer of g in G is as small as possible. When $G = GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$, G' is the set of matrices in G with n distinct eigenvalues.

Let $H_1, ..., H_k$ denote a complete set of representatives for the conjugacy classes of Cartan subgroups of G, and let $H_i' = H_i \cap G', 1 \leq i \leq k$. Then G' is the disjoint union of the sets $G_i' = \{xhx^{-1} : x \in G, h \in H_i'\}, 1 \leq i \leq k$. Further, if H is a Cartan subgroup of G and $h, h' \in H$, then h and h' are conjugate in G just in case they are conjugate via $W(G, H) = N_G(H)/H$. The Weyl integral formula says that the Haar measure on G can be decomposed with respect to these conjugacy classes as

$$\int_G f(x)\ dx = \sum_{i=1}^{k} [W(G, H_i)]^{-1} \int_{H_i} |\Delta_G(h)|^2 \int_{G/H_i} f(xhx^{-1})\ d(xH_i)\ dh,\ f \in C_c^\infty(G).$$

Here for each Cartan subgroup H, $[W(G, H)]$ denotes the number of elements of $W(G, H)$, dh is Haar measure on H, and $d(xH)$ is a G-invariant measure on the quotient space G/H. The function $\Delta_G(h), h \in H$, is called the Weyl denominator. It has the property that $H' = H \cap G' = \{h \in H : \Delta_G(h) \neq 0\}$.

On a Lie group we can differentiate as well as integrate. In our most basic example, when $G = \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, the characters of irreducible unitary representations are the unitary characters $e_n(x) = e^{inx}$. In addition to being continuous group homomorphisms from $\mathbf{T}$ to the circle group S^1, they are eigenfunctions for the differential operator d/dx. In general, the characters $\Theta_\pi, \pi \in \hat{G}$, are eigenfunctions for the abelian algebra $D(G)$ of differential operators on G which commute with both left and right translations by the group. By analysing how these differential operators behave on class functions under restriction to Cartan subgroups, Harish-Chandra showed that if H is a Cartan subgroup of G, then

$$\Theta_\pi(h) = \Delta_G(h)^{-1} \Psi_\pi(h), h \in H'.$$

The function Δ_G is the Weyl denominator which appears in the Jacobean factor of the Weyl integral formula. The numerator Ψ_π has the property that its restriction to any connected component of H' is a finite linear combination of one-dimensional (not necessary unitary) characters of the group H. Which characters can occur is determined by the eigenvalues by which $D(G)$ acts on Θ_π.

Let L be reductive subgroup of G with the property that Cartan subgroups of L are Cartan subgroups of G. Then we can sometimes transfer characters from L to G as follows. For $g \in G$ we define

$$\mathcal{O}_G(g) = \{xgx^{-1} : x \in G\}.$$

It is the conjugacy class of g in G. Then for any $g \in G', \mathcal{O}_G(g) \cap L$ is the finite disjoint union of L-conjugacy classes. Let $X_L(g)$ denote a complete set of representatives for these L-orbits, and let Θ_L be a character of L. Then we transfer the character from L to G by

$$(\mathrm{Tr}_L^G \Theta_L)(g) = \sum_{x \in X_L(g)} \Delta_G(x)^{-1} \Delta_L(x) \Theta_L(x), \quad g \in G'.$$

Here Δ_G and Δ_L are the Weyl denominators for G and L respectively. Note that we just adjust Weyl denominators and then sum over all L-conjugacy classes contained in a given G-conjugacy classin order to obtain a class function on G. There is no guarantee that $\mathrm{Tr}_L^G \Theta_L$ is a character on G. However, it is a class function on G' with the correct Weyl denominator, and so is a possible candidate for a character. Many characters on G can be obtained in this way.

Example 1. Let G be a compact Lie group and let B be a Cartan subgroup of G. Then every $\pi \in \hat{G}$ is a discrete series representation and is parameterized by a character $\chi \in \hat{B}'$. Let π_χ denote the irreducible representation of G corresponding to χ. Then the Weyl character formula says that

$$\Theta_\pi = \mathrm{Tr}_B^G \chi.$$

Example 2. Let L be a Levi subgroup of G. Let $\pi = \mathrm{Ind}_P^G(\sigma)$ where $\sigma \in \hat{L}$ and let $\Theta_\pi, \Theta_\sigma$ denote the characters of π and σ respectively. Then the usual formula for the character of an induced representation says that

$$\Theta_\pi = \mathrm{Tr}_L^G(\Theta_\sigma).$$

This formula for the character of induced representations reduces the problem of understanding the characters of all tempered representations to that of the discrete series characters. These discrete series characters have a very simple formula on compact Cartan subgroups B similar to the Weyl character formula for compact groups. There is an inductive procedure for determining them on other Cartan subgroups $H = TA$.