Representation Theory I

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In this lecture I will give an elementary introduction to the representation theory of groups from the point of view of harmonic analysis.

§**1. Introduction.** Originally, for example in Burnside's classic "Theory of Groups of Finite Order" published in 1897, a group was defined to be a group of transformations of some set, for example a group of permutations of a finite set, a group of symmetries of some geometric object, or a group of linear transformations of a vector space. Today we think of a group as an abstract object, a set together with an operation satisfying certain axioms, and by a group representation we mean a homomorphism from the abstract group to a group of transformations. Thus one abstract group can be represented in many different ways. We will be most interested in representations of G as a group of linear transformations of a vector space.

In this context, the most general definition of a group representation is as follows. Let G be a group and let V be a complex vector space. Let $GL(V)$ denote the group of invertible linear transformations $T : V \to V$ with the operation of composition. (We need to take invertible linear transformations to get a group.) Then a representation of G on V is a homomorphism $\pi : G \to GL(V)$. That is, for each $g \in G, \pi(g)$ is a linear transformation of V and $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$. We say (π, V) or simply π is a representation of G.

One way to get group representations is as follows. Suppose G is a group and X is a set. A (right) group action by G on X is a mapping $X \times G \to X$ denoted by $(x, g) \to x \cdot g$ such that:

(i) if e is the identity element of G, then $x \cdot e = x$ for all $x \in X$;

(ii) for any $g, h \in G, x \cdot (gh) = (x \cdot g) \cdot h$.

Let $V = V(X, \mathbf{C})$ be the complex vector space of functions $f : X \to \mathbf{C}$ with pointwise addition and scalar multiplication. For $f \in V, g \in G, x \in X$, define $T(g)f \in V$ by $[T(g)f](x) = f(x \cdot g)$. It is easy to check that $T(g) : V \to V$ is a linear transformation, and that $T(e) = I$ is the identity map. Further, for all $f \in V, g, h \in G, x \in X$.

$$[T(gh)f](x) = f(x \cdot (gh)) = f((x \cdot g) \cdot h) = [T(h)f](x \cdot g) = [T(g)T(h)f](x).$$

That is $T(gh) = T(g)T(h)$ is the composition of $T(g)$ and $T(h)$. Thus $T(g) \in GL(V)$ is invertible (with inverse $T(g^{-1})$), and $T : G \to GL(V)$ is a group homomorphism.

Example 1. Let $G = S_n$ be the group of permutations of the set $X = \{1, 2, 3, ..., n\}$. Then $(i, \sigma) \to i\sigma, i \in X, \sigma \in G$, is a group action. In this case $V(X, \mathbf{C}) \simeq \mathbf{C}^\mathbf{n}$ is an n-dimensional vector space with basis $\{f_1, ..., f_n\}$ defined by

$$f_i(j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

For each permutation $\sigma \in S_n$, $T(\sigma)$ is the linear transformation such that
$T(\sigma) : f_i \to f_{i\sigma^{-1}}$.

Example 2. Let G be any group and let $X = G$. Then $(x, g) \to xg, x, g \in G$, is a group action of G on itself. Now if $V = V(G, \mathbf{C})$, we have a representation of G on $GL(V)$ given by $R(g)f(x) = f(xg)$. It is usually called the right regular representation of G. If G is a big group, such as the set $\mathbf{R}$ of real numbers with addition, the vector space $V(\mathbf{R}, \mathbf{C})$ of **all** complex-valued functions on G is ridiculously big. To get something more reasonable, we add more structure to our groups, and take smaller spaces of functions on G.

§2. Topological Groups.

A topological space is a set X with a collection $\mathcal{T} = \mathcal{T}(X)$ of subsets of X called open sets satisfying some natural axioms:

(i) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;

(ii) $O_1, O_2 \in \mathcal{T}$ implies that $O_1 \cap O_2 \in \mathcal{T}$;

(iii) $O_\alpha \in \mathcal{T}, \alpha \in A$, implies that $\cup_{\alpha \in A} O_\alpha \in \mathcal{T}$, where A is any index set.

A subset K of X is called compact if every open cover of K has a finite subcover.

For example, the real numbers $\mathbf{R}$ and complex numbers $\mathbf{C}$ are examples of topological spaces with the usual notion of open sets. So are the vector spaces $\mathbf{R}^n$ and $\mathbf{C}^n$. Compact subsets of these spaces are exactly the closed and bounded sets.

If X and Y are topological spaces, we say $f : X \to Y$ is continuous if $f^{-1}(O) = \{x \in X : f(x) \in O\}$ is an open set of X for every open set O of Y. In the special case that $Y = \mathbf{C}$ with the usual topology, we write $C(X)$ for the set of continuous mappings from X to $\mathbf{C}$. Note that $C(X)$ is a complex vector space with pointwise addition and scalar multiplication of functions.

A topological group is a topological space G which is also a group for which the group operations are continuous. That is, the mappings $m : G \times G \to G$ given by $m(x, y) = xy$ and $i : G \to G$ given by $i(x) = x^{-1}$ are continuous. ($G \times G$ has a natural topology called the product topology.) Topological groups are also usually assumed to be Hausdorff. This means that for any distinct points $x, y \in G$ there are disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$.

Examples of topological groups are $\mathbf{R}$, the group of real numbers with addition and $\mathbf{R}^{\times} = \mathbf{R} - \{0\}$, the group of non-zero real numbers with multiplication. These are both non-compact abelian groups. Another important example is the quotient group $\mathbf{T} = \mathbf{R}/\mathbf{2\pi Z}$ where $\mathbf{R}$ is the additive group of real numbers and $2\pi\mathbf{Z}$ is the subgroup of $\mathbf{R}$ consisting of all integer multiples of 2π. As a topological space it is the interval $[0, 2\pi]$ with the endpoints identified, hence a circle. It is a compact abelian group. Continuous functions on $\mathbf{T}$ can be thought of as continuous functions on the interval $[0, 2\pi]$ with $f(0) = f(2\pi)$ or as continuous 2π-periodic functions on the real line.

To get lots of examples of non-abelian topological groups we look at groups of matrices. For $n \geq 1$, let $GL(n, \mathbf{R})$ and $GL(n, \mathbf{C})$ denote the groups of invertible $n \times n$ matrices with real or complex entries. They have natural topologies as open subsets of $\mathbf{R}^{n^2}$ or $\mathbf{C}^{n^2}$. Any closed subgroup of $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$ is called a linear Lie group. For example, my favorite Lie group is $SL(2, \mathbf{R}) = \{X \in GL(2, \mathbf{R}) : \det X = 1\}$.

§3. Hilbert Spaces.

Let W be a complex vector space. W is called a Hilbert space if it has the following extra structure. First, W has a Hermitian inner product that assigns to each $u, v \in W$ a complex number $< v, w >$ such that:

(i) $< a_1 v_1 + a_2 v_2, w >= a_1 < v_1, w > + a_2 < v_2, w >, \quad a_1, a_2 \in \mathbf{C}, v_1, v_2, w \in W$;

(ii) $< v, w >= \overline{< w, v >}, \quad v, w \in W$;

(iii) $< v, v > \geq 0$ for all $v \in W$ and $< v, v >= 0$ if and only if $v = 0$.

Given a Hermitian inner product on W we can define a norm on W by

$$\|v\| = (< v, v >)^{\frac{1}{2}}, \ v \in W.$$

Once we have a norm we can talk about convergent sequences. If v_n is a sequence in W, we say v_n is Cauchy if $\lim \|v_n - v_m\| = 0$ as $n, m \to \infty$. We say v_n converges to $v \in W$ if $\lim \|v_n - v\| = 0$ as $n \to \infty$. Now the final requirement for a Hilbert space is that it be complete for this norm. That is, every Cauchy sequence in W converges to an element of W.

Example 1. For $n \geq 1$, $W = \mathbf{C}^n$, is a Hilbert space with the standard inner product. If $v = (v_1, ..., v_n)$ and $w = (w_1, ..., w_n)$, the inner product and associated norm are given by

$$< v, w >= \sum_{i=1}^{n} v_i \ \overline{w}_i, \quad \|v\| = (\sum |v_i|^2)^{\frac{1}{2}}.$$

Example 2. Let $\mathbf{T} = \mathbf{R}/\mathbf{2\pi Z}$ as above. We can define a Hermitian inner product and norm on the complex vector space $C(\mathbf{T})$ by

$$< f, g >= \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx, \quad \|f\| = (\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx)^{\frac{1}{2}}.$$

It is analogous to the example above, but with an integral (continuous sum) instead of finite sum. The factor of $1/2\pi$ is used so that $\|1\| = 1$, where 1 denotes the function that is identically one. However, $C(\mathbf{T})$ is not complete with this norm. To get a Hilbert space we need to complete $C(\mathbf{T})$ to $L^2(\mathbf{T})$, the space of measurable complex-valued functions f such that

$$\|f\| = (\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx)^{\frac{1}{2}} < \infty.$$

Let W be a Hilbert space. An invertible linear transformation $T : W \to W$ is called unitary if $< Tv, Tw >=< v, w >$ for all $v, w \in W$. The unitary operators on W with the operation of composition form a group which we will call $U(W)$. If $W = \mathbf{C}^n$ as above, then $U(W)$ can be identified with the group $U(n) = \{X \in GL(n, \mathbf{C}) : \overline{X}^t X = I\}$, the compact group of $n \times n$ unitary matrices. In particular, $U(1) = \{z \in \mathbf{C} : |z|^2 = 1\} = \{e^{it} : t \in \mathbf{R}\}$ is the circle group.

§4. Unitary Representations.

Let G be a topological group and let W be a Hilbert space. A unitary representation of G on W is a group homomorphism $\pi : G \to U(W)$ which is continuous in the following sense. Since W has an inner product, for every $v, w \in W$, we can define a complex-valued function f on G, called a matrix coefficient of π, by $f(x) =< \pi(x)v, w >, x \in G$. Now π is called continuous if f is continuous for every choice of $v, w \in W$.

Suppose that $\dim W = 1$. In this case $U(W)$ is isomorphic to the circle group $S^1 = \{z \in \mathbf{C} : |z| = 1\}$, and using this identification, $\pi : G \to S^1$ can be thought of as a continuous group homomorphism from G to S^1. One-dimensional unitary representations are called unitary characters. Every group has at least one unitary character, the trivial character $\chi(g) = 1$ for all $g \in G$.

Example. Let $G = \mathbf{R}$. Then for all $y \in \mathbf{R}, e_y(x) = e^{iyx}$ is a unitary character of $\mathbf{R}$ since $e_y(x_1 + x_2) = e^{iy(x_1+x_2)} = e^{iyx_1}e^{iyx_2} = e_y(x_1)e_y(x_2),\ x_1, x_2 \in \mathbf{R}$. Further, if $y = n \in \mathbf{Z}$, then $e_n(x + 2\pi) = e_n(x)e^{2\pi in} = e_n(x)$ so that e_n factors to give a unitary character of $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$.

Let (π, W) be a unitary representation of G. A closed subspace $V \subset W$ is called invariant if $\pi(g)v \in V$ for all $g \in G, v \in V$. In this case we obtain a unitary representation of G on V by restricting the operators $\pi(g), g \in G$, to V. It is called a subrepresentation of π. If V is invariant, let

$$V^\perp = \{w \in W :< v, w >= 0 \ \forall v \in V\}.$$

Then for $w \in V^\perp, v \in V, g \in G$,

$$< \pi(g)w, v >=< \pi(g)w, \pi(g)\pi(g^{-1})v >=< w, \pi(g^{-1})v >= 0$$

since $\pi(g^{-1})v \in V$ and $w \in V^\perp$. Thus $\pi(g)w \in V^\perp$ and so $V^\perp$ is also an invariant subspace of W. Further, $W = V \oplus V^\perp$, and we can regard (π, W) as the direct sum of representations of G on V and $V^\perp$. Now (π, W) is called irreducible if it has no proper invariant subspaces. For example, if W is one-dimensional, (π, W) is irreducible since W has no proper subspaces. If G is abelian, these are the only irreducible representations. However nonabelian groups have higher dimensional, and even infinite-dimensional irreducible representations.

Suppose that (π_1, W_1) and (π_2, W_2) are two unitary representations of G. We say that (π_1, W_1) and (π_2, W_2) are (unitarily) equivalent if there is an invertible linear operator $T : W_1 \to W_2$ such that $< Tv, Tw >=< v, w >$ for all $v, w \in W_1$ and

$$\pi_2(g) = T\pi_1(g)T^{-1} \ \forall \ g \in G.$$

We regard equivalent representations as being the same.

Two of the most important problems in representation theory are the following.

(1) Given a topological group G, find $\hat{G}$, the set of equivalence classes of irreducible unitary representations.

(2) Given an arbitary unitary representation of G, describe how to decompose it into irreducible constituents.

§5. Harmonic Analysis.

The theory of harmonic analysis on groups originated in the eighteenth century with the problem of representing an arbitrary periodic function by a trigonometric series. In its modern version, we think of periodic functions as functions on the group $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Instead of expanding real-valued functions on $\mathbf{T}$ in terms of the functions $\sin nx, \cos nx$, $n = 0, 1, 2, ...$, it is more convenient to expand complex-valued functions on $\mathbf{T}$ in terms of the complex exponential functions $e_n(x) = e^{inx} = \cos nx + i \sin nx, n \in \mathbf{Z}$.

Recall we have the Hilbert space $L^2(\mathbf{T})$ which was obtained by completing $C(\mathbf{T})$. For $n, m \in \mathbf{Z}$,

$$< e_n, e_m >= \frac{1}{2\pi} \int_0^{2\pi} e^{inx} e^{-imx} \, dx = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{otherwise.} \end{cases} .$$

Thus $\{e_n\}$ is an orthonormal set in $L^2(\mathbf{T})$. The theory of Fourier series says that it is a complete orthonormal set, that is a Hilbert space basis for $L^2(\mathbf{T})$. That is, for every $f \in L^2(\mathbf{T})$, we can expand f as a Fourier series $f = \sum_n \hat{f}(n) e_n$ where $\hat{f}(n) =< f, e_n >$ is called the n^{th} Fourier coefficient of f. Here the series does not necessarily converge pointwise for each $x \in \mathbf{T}$, but the partial sums of the Fourier series converge to f in the L^2 norm. What does this have to do with representation theory?

As we saw above, the functions e_n used in the Fourier series are unitary characters of $\mathbf{T}$. In fact $\hat{\mathbf{T}} = \{e_n : n \in \mathbf{Z}\}$. This is the solution of problem (1) in this case.

Recall we have the regular representation of $\mathbf{T}$ on $L^2(\mathbf{T})$ given by $R(y)f(x) = f(x+y)$. For $f, g \in L^2(\mathbf{T}), y \in \mathbf{T}$, using the change of variables $u = x + y, du = dx$, we have

$$< R(y)f, R(y)g >= \frac{1}{2\pi} \int_0^{2\pi} f(x+y)\overline{g(x+y)} \, dx = \frac{1}{2\pi} \int_y^{y+2\pi} f(u)\overline{g(u)} \, du =< f, g >$$

since f, g are 2π-periodic, so that the integral over any interval of length 2π is the same. Thus $R : \mathbf{T} \to U(L^2(\mathbf{T}))$ is a unitary representation.

For each $n \in \mathbf{Z}$, $\mathbf{C}e_n$ is a 1-dimensional subspace of $L^2(\mathbf{T})$. For $y \in \mathbf{T}, R(y)e_n(x) = e_n(x+y) = e_n(x)e_n(y) = e^{iny}e_n(x)$; that is $R(y)e_n = e^{iny}e_n \in \mathbf{C}e_n$. Thus $\mathbf{C}e_n$ is an invariant subspace of $L^2(\mathbf{T})$. Since it is one-dimensional, it is irreducible. Now, by the theory of Fourier series,

$$L^2(\mathbf{T}) = \oplus \sum_{n \in \mathbf{Z}} \mathbf{C}e_n$$

is the decomposition of $L^2(\mathbf{T})$ into irreducible subspaces.

In order to generalize this example, we need to be able to integrate functions on our groups G. Assume now that G is a locally compact topological group. Then there is a unique (up to constant) right Haar measure on G. This means we can define integration on G so that for every $g \in G$ and integrable function f,

$$\int_G f(x) \, dx = \int_G f(xg) \, dx = \int_G [R(g)f](x) \, dx$$

where

$$[R(g)f](x) = f(xg), x, g \in G.$$

That is our integral is invariant under translation on the right. (Since G may not be abelian we need to distinguish between right and left Haar measures.)

Once we have (right) Haar measure on G, we can define $L^2(G)$ to be the complex vector space of measurable functions $f : G \to \mathbf{C}$ such that

$$\|f\| = (\int_G |f(x)|^2 dx)^{\frac{1}{2}} < \infty.$$

It is a Hilbert space with the inner product

$$< f, g >= \int_G f(x)\overline{g(x)} dx, \qquad f, g \in L^2(G).$$

As in the example, because we use a translation invariant measure, the right regular representation $R : G \to U(L^2(G))$ is a a unitary representation of G on $L^2(G)$. Now L^2 harmonic analysis on G is the decomposition of the regular representation of G on $L^2(G)$ into irreducible representations.

Suppose that G is a compact group, like $\mathbf{T}$, but is not necessarily abelian. Then every $(\pi, W) \in \hat{G}$ is finite-dimensional. The dimension d_π of W is called the degree of π. We define a matrix coefficient of $(\pi, W) \in \hat{G}$ to be any function of the form

$$\phi_{v,w}(x) =< \pi(x)v, w >, v, w \in W.$$

These are all continuous functions by our continuity requirement on π, and G is compact, so every matrix coefficient is in $L^2(G)$. Let $L^2(G)_\pi$ denote the subspace of $L^2(G)$ spanned by the matrix coefficients of $\pi \in \hat{G}$. It is an invariant subspace of dimension d_π^2 and the restriction of the right regular representation of G to $L^2(G)_\pi$ can be decomposed as the direct sum of d_π irreducible subrepresentations, each of which is equivalent to π. Functions in $L^2(G)_\pi$ are regarded as elementary functions because they transform in the simplest possible way with respect to translations by elements of the group. The Peter-Weyl theorem says that

$$L^2(G) = \oplus \sum_{\pi \in \hat{G}} L^2(G)_\pi.$$

That is, $L^2(G)$ decomposes as the direct sum of irreducible subspaces, and each $\pi \in \hat{G}$ occurs with multiplicity d_π.

When G is not compact, we can no longer decompose $L^2(G)$ as the direct sum of irreducible subspaces. For example, suppose that $G = \mathbf{R}$ is the additive group of real numbers. Since G is abelian, all irreducible unitary representations are one-dimensional, that is are unitary characters. For each $y \in \mathbf{R}, e_y(x) = e^{ixy}, x \in \mathbf{R}$, is a unitary character of $\mathbf{R}$, and all unitary characters are of this form. Thus $\hat{G} \simeq \mathbf{R} \simeq G$ so that G is self-dual. However the characters are not L^2 functions since for any $y \in \mathbf{R}$,

$$\int_{\mathbf{R}} |e^{ixy}|^2 \, dx = \int_{\mathbf{R}} 1 \, dx = \infty.$$

Thus $L^2(\mathbf{R})$ has no irreducible invariant subspaces.

If $f \in L^2(\mathbf{R})$ is integrable, we can define the Fourier transform

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-inx} dx, \quad y \in \mathbf{R}.$$

Now if we define partial Fourier integrals

$$s_N(x) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \hat{f}(y) e^{iny} dy, \quad x \in \mathbf{R}, \quad N = 1, 2, 3, ...,$$

then s_N converges to f in $L^2(\mathbf{R})$ as $N \to \infty$. Thus we can think of $f \in L^2(\mathbf{R})$ as being equal to its formal Fourier integral

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e_y dy,$$

and we think of

$$L^2(\mathbf{R}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{C} e_y \; dy$$

as a direct integral of irreducible unitary representations.

It is the non-compact, non-abelian groups that really get interesting. For example, let $G = SL(2, \mathbf{R})$ be the group of 2×2 real matrices with determinant one. The only finite dimensional unitary representation of G is the trivial representation. All other irreducible unitary representations are infinite-dimensional. Some of these, the discrete series, have matrix coefficients which are L^2 functions, and so, as in the case of compact groups, give invariant subspaces of $L^2(G)$. The sum of these subspaces is an invariant subspace $L^2(G)_d$. Others, the principal series do not have L^2 matrix coefficients, but part of $L^2(G)$, the orthogonal complement of $L^2(G)_d$, is a direct integral of these representations. Finally, there are others, called complementary series, which do not contribute at all to $L^2(G)$.