The Gröbner Fan and Gröbner Walk for Submodules

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Gröbner bases are one of the primary tools of computational commutative algebra.
Why?

- computable,
- used in many algorithms:
  - ideal/submodule membership problem
  - syzygies and minimal free resolutions
  - etc.

At its essence, this dissertation is about classifying and computing Gröbner bases for submodules of free modules over polynomial rings.

Gröbner Fan (ideal case by Mora and Robbiano, 1988)

- enumerates all reduced marked Gröbner bases

- identifies each reduced marked Gröbner basis with a region of $\mathbb{R}^n$

- allows interplay between commutative algebra and polyhedral geometry

Gröbner Walk (ideal case by Collart, Kalkbrener, and Mall, 1993)

- algorithm for converting from one reduced marked Gröbner basis to another
- uses Gröbner fan
- useful in computing Gröbner bases for elimination term orders or other inefficient term orders

State Polytope (ideal case by Bayer and Morrison, 1988)

- related to the Gröbner fan for a homogeneous ideal/submodule
- allows interplay of polyhedral geometry and commutative algebra

Term Orders

Let $R = k[x_1, \ldots, x_n]$, k a field.
Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{Z}^+)^n$

Terms of R: $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Denote the set as `Terms`$(R)$.

Examples of term orders:

lexicographic order is given by:

$X^\alpha > X^\beta$ if and only if there exists $0 \leq i < n$ such that $\alpha_j = \beta_j$, for $1 \leq j \leq i$, and $\alpha_{j+1} > \beta_{j+1}$.

degree lexicographic is given by:

$X^\alpha > X^\beta$ if and only if either $\sum \alpha_i > \sum \beta_i$ or both $\sum \alpha_i = \sum \beta_i$ and $X^\alpha > X^\beta$ in lexicographic order.

Classification of Term Orders

A term order is classified by an element of $\mathtt{Mat}_{m\times n}(\mathbb{R})$.

Such a matrix with rows $(u_1, \ldots, u_m)$ defines a term order by:

$$X^\alpha > X^\beta \text{ if and only if } (\alpha \cdot u_1, \ldots, \alpha \cdot u_m) >_{\mathtt{lex}} (\beta \cdot u_1, \ldots, \beta \cdot u_m),$$

where $>_{\mathtt{lex}}$ means “greater than” in the lexicographic order.

(Term orders were originally classified by Riquier (1910), Kolchin (1973), Trevisan (1953), and Zaĭceva (1953). Most recently, the clasification was rediscovered by Robbiano (1985).)

Monomial Orders

Consider R^t, the free module on R with t components. Let

$$\begin{array}{rcl} \mathbf{e}_1 & = & (1, 0, 0, \ldots, 0) \\ \mathbf{e}_2 & = & (0, 1, 0, \ldots, 0) \\ \vdots & & \vdots \\ \mathbf{e}_t & = & (0, 0, \ldots, 0, 1) \end{array}$$

be the standard basis vectors on R^t.

Monomials of R^t: $\mathbf{X} = X^\alpha \mathbf{e}_i = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mathbf{e}_i$, $1 \leq i \leq t$. Denote the set as $\mathtt{Mon}(R^t)$.

Examples of monomial orders:

term over position with a lexicographic order is given by:
$X^\alpha \mathbf{e}_i > X^\beta \mathbf{e}_j$ if and only if either $X^\alpha > X^\beta$ in lexicographic order or both $\alpha = \beta$ and $i < j$.

position over term with a degree lexicographic order is given by:
$X^\alpha > X^\beta$ if and only if either $i > j$ or both $i = j$ and $X^\alpha > X^\beta$ in degree lexicographic order.

Leading Monomials

Definition:

Let $\mathbf{f} \in R^t$ and $\mathtt{Mon}(\mathbf{f}) = \{\text{monomials in } \mathbf{f}\}$.
Let $>$ be a monomial order on R^t.
Define $\mathtt{lm}_>(\mathbf{f}) = \mathbf{X}$ such that $\mathbf{X} > \mathbf{Y}\ \forall$
$\mathbf{X} \neq \mathbf{Y} \in \mathtt{Mon}(\mathbf{f})$.
Let $S \subseteq R^t$.
Define $\mathtt{lm}_>(S) = \{\mathtt{lm}_>(\mathbf{f}) \,|\, \mathbf{f} \in S\}$.

Example: Let $>$ be a term over position order with $x > y$ lexicographic on each component.

$$\mathtt{lm}_> \left(x^2\mathbf{e}_1 + (x^3 - 3yx^2 - 4y^3)\mathbf{e}_2 \right) = x^3\mathbf{e}_2.$$

Gröbner Bases

Definition: Let $M \subseteq R^t$ be a submodule. Let $>$ be a monomial order. A set

$$G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_a\} \subseteq M$$

is a **Gröbner basis** for M with respect to $>$ if

$$\langle \mathtt{lm}_>(G) \rangle = \langle \mathtt{lm}_>(M) \rangle .$$

• G is **reduced** if $\forall \mathbf{g} \in G$, there does not exist $\mathbf{X} \in \mathtt{Mon}(\mathbf{g})$ and $\mathbf{f} \in G \setminus \{\mathbf{g}\}$ such that $\mathtt{lm}_>(\mathbf{f})$ divides $\mathbf{X}$.
• G is **marked** if for each $\mathbf{g} \in G$, the monomial $\mathtt{lm}_>(\mathbf{g})$ is identified.

Theorem: For any submodule $M \subseteq R^t$, there are only finitely many reduced marked Gröbner bases.

Polyhedral Geometry

A **polyhedron** is the intersection of finitely many closed half spaces in $\mathbb{R}^a$.
If each of the *supporting* hyperplanes of the polyhedron P intersects the origin, then the polyhedron is a **(polyhedral) cone**.
A polyhedron that is bounded is a **polytope**.

A **face** is a subset of a polyhedron that maximizes some linear functional.

A **fan** is a collection of cones for which the intersection of any two cones is a face in each cones.

The **support** of a polyhedron [resp. fan] is the set of points in $\mathbb{R}^a$ contained in the polyhedron [resp. fan].

Gröbner Fans

$$
\begin{array}{ccc}
 & \left\{\begin{array}{c}\text{monomial}\\ \text{orders}\\ \text{as represented}\\ \text{by Rust \& Reid}\end{array}\right\} & \\
\swarrow \varphi & & \kappa \searrow \\
\left\{\begin{array}{c}\text{gradings}\\ \text{of the}\\ \text{free}\\ \text{module}\end{array}\right\} & \leftarrow \textit{compatible} \rightarrow & \left\{\begin{array}{c}\text{reduced}\\ \text{marked}\\ \text{Gröbner}\\ \text{bases}\\ \text{of } M\end{array}\right\} \\
\downarrow \lambda & & \downarrow \mu \\
\left\{\begin{array}{c}\text{support}\\ \text{of the}\\ \text{Gröbner}\\ \text{fan of } M\end{array}\right\} & \leftarrow \textit{support} \rightarrow & \left\{\begin{array}{c}\text{cones}\\ \text{of the}\\ \text{Gröbner}\\ \text{fan of } M\end{array}\right\}
\end{array}
$$

$$
\kappa : \; > \mapsto \left\{\begin{array}{l}\text{reduced marked}\\ \text{Gröbner basis}\\ \text{with respect to } >\end{array}\right.
$$

Classification of Monomial Orders

Rust and Reid

Any monomial order $>$ on R^t can be given by the following:

- matrices U_i, $1 \leq i \leq t$ *- term order on component*
- vectors γ_i, $1 \leq i \leq t$ *- weight of component*
- a non-negative integer matrix $\{t_{ij}\}$, $1 \leq i, j \leq t$, (with restrictions) *- # of rows compared*
- an element $\sigma \in S_t$, (with restrictions) *- breaks ties*

Let $\mathtt{Pr}_d(\alpha)$ be the projection onto the first d coordinates of the vector α. Then $X^\alpha \mathbf{e}_i > X^\beta \mathbf{e}_j$ if an only if either

- $\mathtt{Pr}_{t_{ij}}(U_i\alpha + \gamma_i) >_{\mathtt{lex}} \mathtt{Pr}_{t_{ij}}(U_j\beta + \gamma_j)$, or
- $\mathtt{Pr}_{t_{ij}}(U_i\alpha + \gamma_i) = \mathtt{Pr}_{t_{ij}}(U_j\beta + \gamma_j)$ and $\sigma(i) > \sigma(j)$.

Generalizing POT

If $t_{ij} = 0$ and $\sigma(i) > \sigma(j)$, then $X\mathbf{e}_i > Y\mathbf{e}_j$ for any terms $X, Y \in R$.
- This is a POT-like property.

Create an equivalence class $\sim$ on $\{1, 2, \ldots, t\}$ by $i \sim j$ if and only if $t_{ij} \neq 0$.
- This equivalence class is independent of choice of matrices, vectors, etc. representing the monomial order $>$.
- Call $([h_1], [h_2], \ldots, [h_q])$ the **type** of the monomial order, where $1 \leq q \leq t$, $1 \leq h_1, \ldots, h_q \leq t$ are representatives of each equivalence class, and $\sigma(h_1) > \sigma(h_2) > \cdots > \sigma(h_q)$.

For each type $([h_1], [h_2], \ldots, [h_q])$ of monomial orders, there is a separate Gröbner fan, which has support $\mathtt{Mat}_{q \times n}(\mathbb{R}^+) \times \mathbb{R}^t$.

Gradings and Map λ

Let $P = ([h_1], [h_2], \ldots, [h_q])$ be an ordered partition of $\{1, \ldots, t\}$, $1 \leq q \leq t$.
Let $N \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+)$, with rows $n_1, \ldots, n_q$.
Let $r = (r_1, \ldots, r_t) \in \mathbb{R}^t$.
Let $\tau = (N, r, P)$.

N-grading on $\mathtt{Terms}(R)$: $X^\alpha \mapsto N\alpha \in \mathbb{R}^q$

τ-grading on $\mathtt{Mon}(R^t)$:

$$X^\alpha \mathbf{e}_i \stackrel{\tau}{\mapsto} (\underbrace{-\infty, \ldots, -\infty}_{j-1 \text{ copies}}, \alpha \cdot n_j + r_i, \underbrace{-\infty, \ldots, -\infty}_{q-j \text{ copies}})$$

$\in (\mathbb{R} \cup \{-\infty\})^q$, where $i \in [h_j]$.

Map λ: an (N, r, P)-grading is mapped to a point $(N, r) \in \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$ in the support of the Gröbner fan for monomial orders of type P.

Map φ

$$\left\{\begin{array}{c}\text{monomial}\\ \text{orders as}\\ \text{represented}\\ \text{by}\\ \text{Rust \& Reid}\end{array}\right\} \xrightarrow{\varphi} \left\{\begin{array}{c}\text{gradings}\\ \text{of } R^t\end{array}\right\} \xrightarrow{\lambda} \left\{\begin{array}{c}\text{support}\\ \text{of the}\\ \text{Gröbner}\\ \text{fan of } M\end{array}\right\}$$

For each type $P = ([h_1], [h_2], \ldots, [h_q])$, the map φ is given by :

$$\left\{\begin{array}{c}\text{monomial order of}\\ \text{type } P \text{ given by}\\ \text{matrices } U_1, \ldots, U_t,\\ \text{vectors } \gamma_1, \ldots, \gamma_t,\\ \text{etc.}\end{array}\right\} \overset{\varphi}{\mapsto} (N, r, P) - \text{grading},$$

where N is a $q \times n$ matrix of rows $(v_1, \ldots, v_q)$, where row v_i is the first row of the matrix U_{h_i} for $1 \leq i \leq q$,
and $r = (r_1, \ldots, r_t)$ is a vector with r_i the first coordinate of the vector γ_i for $1 \leq i \leq t$.

Compatibility

$$\left\{\begin{array}{c}\text{gradings}\\ \text{of the}\\ \text{free}\\ \text{module}\end{array}\right\} \leftarrow \textit{compatible} \rightarrow \left\{\begin{array}{c}\text{reduced}\\ \text{marked}\\ \text{Gröbner}\\ \text{bases}\\ \text{of } M\end{array}\right\}$$

A set of marked vectors $G \subseteq R^t$ is **compatible** with a τ-grading of R^t if and only if $\forall\ \mathbf{g} \in G$ with a marked leading monomial $\mathbf{X}$,

$$\tau(\mathbf{X}) \geq_{\mathtt{lex}} \tau(\mathbf{Y}), \forall\ \mathbf{Y} \in \mathtt{Mon}(\mathbf{g}).$$

For any monomial order $>$, the $\varphi(>)$-grading is compatible with the Gröbner basis $\kappa(>)$ because it is compatible with every vector in R^t that is marked by $>$.

Cones of the Fan and Map μ

$$\left\{\begin{array}{c}\text{reduced}\\ \text{marked}\\ \text{Gröbner}\\ \text{bases}\\ \text{of } M\end{array}\right\} \xrightarrow{\mu} \left\{\begin{array}{c}\text{cones}\\ \text{of the}\\ \text{Gröbner}\\ \text{fan of } M\end{array}\right\}$$

Below, defines the map μ:

Each reduced marked Gröbner basis G for a submodule $M \subseteq R^t$ with respect to some type $P = ([h_1], [h_2], \ldots, [h_q])$ monomial order $>$ is associated with a subset of $\mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$, called a G-**cone**, defined by

$$C_G = \left\{ \begin{array}{c} (N, r) \in \\ \mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t \end{array} \middle| \begin{array}{c} > \text{ is compatible} \\ \text{on } G \text{ with a} \\ (N, r, P)\text{-grading} \end{array} \right\}.$$

The Gröbner Fans

For any submodule $M \subseteq R^t$, the fan generated by the cones

$$\left\{ C_G \;\middle|\; \begin{array}{c} G \text{ is a reduced marked Gröbner} \\ \text{basis for } M \text{ with respect} \\ \text{to a type } P \text{ monomial order} \end{array} \right\}$$

is a fan in the space $\mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$. Call the fan the "$P$ Gröbner fan" for M.

The support of the fan is $\mathtt{Mat}_{q\times n}(\mathbb{R}^+) \times \mathbb{R}^t$.

Proof:

- G-cones are polyhedral cones.
- The intersection of two cones is a face of each.
- Check the support.

Theorem: Let F be the $P = ([h_1], [h_2], \ldots, [h_q])$ Gröbner fan for a submodule $M \subseteq R^t$. Let F_i be the $[h_i]$ Gröbner fan for the submodule

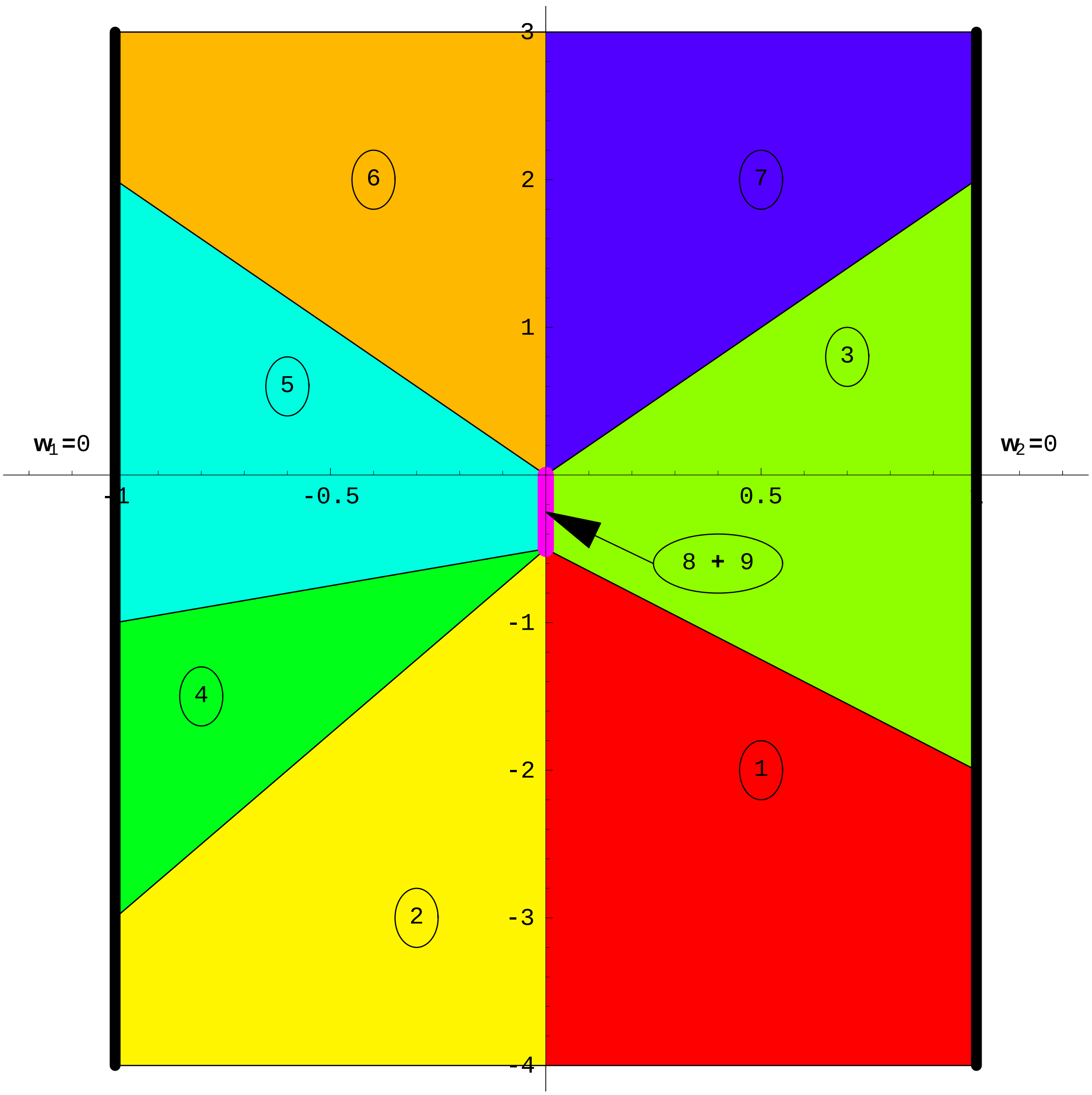
$$N_i = \left\{ \sum_{j \in [h_i]} f_j \mathbf{e}_j \;\middle|\; \begin{array}{c} \sum\limits_{j \in [h_i] \cup \cdots \cup [h_q]} f_j \mathbf{e}_j \in M \\ \text{for some } f_j \in R, \\ \text{for } j \in [h_i] \cup \cdots \cup [h_q] \end{array} \right\} \subseteq R^{|[h_i]|},$$

for $1 \leq i \leq q$. Then $F = \prod_{i=1}^{q} F_i$.

One difference between Gröbner fans for ideals and for submodules is that in the submodule case, the cones corresponding to Gröbner bases may have codimension greater than zero.

I have found an algorithm for computing the Gröbner fans, which includes finding which cones with codimension greater than zero correspond to Gröbner bases.

A Slice of the $\{1,2\}$ Gröbner Fan for $\left\langle \begin{pmatrix} x^2+y^2 \\ x^2 \end{pmatrix}, \begin{pmatrix} y \\ x^2+y^2 \end{pmatrix} \right\rangle$



The Gröbner Walk

The Gröbner walk is a method for converting one Gröbner basis into another Gröbner basis.

Suppose you have the Gröbner basis G_b for the submodule $M \subseteq R^t$ with respect to the monomial order $>_b$. Further, suppose you would like to compute the Gröbner basis for M with respect to a monomial order $>_e$.

If $>_b$ and $>_e$ are both of type P, then each corresponds to a point in the P Gröbner fan for M. The walk follows a path between these points, and each time it crosses from one cone to the next, it computes the Gröbner basis in the new cone.

The following theorem shows how the conversion is done:

Theorem: Let $M \subseteq R^t$ be a submodule. Let there be a grading on R^t defined by ϕ. Let $>_1$ and $>_2$ be monomial orders which are compatible with ϕ, and let G be a Gröbner basis for M with respect to $>_2$. Let H be a Gröbner basis for $\left\langle \mathtt{lm}_\phi(M) \right\rangle$ with respect to $>_1$. Using the division algorithm with respect to $>_2$, write each $\mathbf{h} \in H$ as

$$\mathbf{h} = \sum_{\mathbf{g} \in G} p_{\mathbf{g},\mathbf{h}} \mathtt{lm}_\phi(\mathbf{g}),$$

with $p_{\mathbf{g},\mathbf{h}} \in R$. For each $\mathbf{h} \in H$ define $\mathbf{f_h}$ by

$$\mathbf{f_h} = \sum_{\mathbf{g} \in G} p_{\mathbf{g},\mathbf{h}} \mathbf{g}.$$

Then the set $F = \{\mathbf{f_h} | \mathbf{h} \in H\}$ forms a Gröbner basis for M with respect to $>_1$.

Sketch of the algorithm:

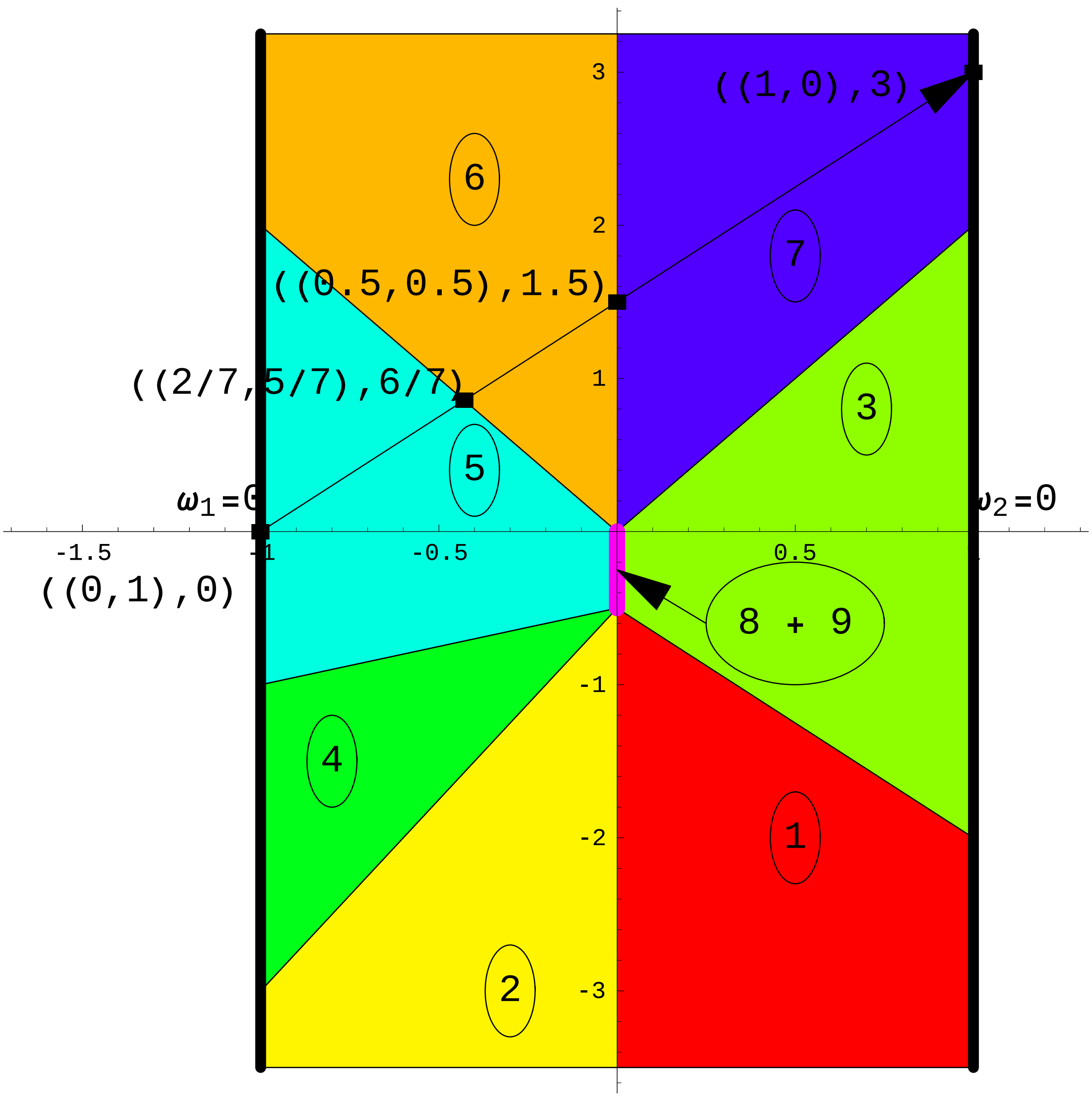
- Let G be the original reduced marked Gröbner basis with respect to $>_b$ of type P. Want Gröbner basis for $>_e$ of type P.

- Let C be the cone for G in the P Gröbner fan.

- Let $v(t)$ be the line from $\lambda \circ \varphi(>_b)$ to $\lambda \circ \varphi(>_e)$, $0 \le t \le 1$.

- Repeat until cone C contains $\lambda \circ \varphi(>_e)$.
  - Let $0 \le t_0 \le 1$ be the largest such that $v(t_0) \in C$.
  - Use previous Theorem to find the new reduced marked Gröbner basis G in next cone.
  - Compute the new cone C for G in the P Gröbner fan.

A Gröbner Walk on the $\{1,2\}$ Gröbner Fan for $\left\langle \begin{pmatrix} x^2+y^2 \\ x^2 \end{pmatrix}, \begin{pmatrix} y \\ x^2+y^2 \end{pmatrix} \right\rangle$



The State Polytope

The state polytope is a polytope that generalizes the Newton Polytope and is closely related to the Gröbner fan for homogeneous submodules. The precise relationship is that the *normal fan* of the state polytope is the P Gröbner fan for the submodule M, where M is (N, r, P)-homogeneous, for some matrix N and vector r.

Definition: For any polyhedron $Q \subseteq \mathbb{R}^t$ and any face F of Q, the **normal cone of F at Q**, denoted $N_Q(F)$, is the closure of the set

$$\{\omega \in \mathbb{R}^t | \forall x \in F, \omega \cdot x \geq \omega \cdot y \ \ \forall y \in Q\}.$$

The collection of normal cones as F ranges over the faces of Q is a fan. This fan is denoted $N(Q)$ and is called the **normal fan of Q**.