(1) Note that the graphs intersect at $x = 0$ and $x = 1$. Then
\[
\text{Volume} = \int_0^1 2\pi x (\sqrt{x} - x^2) \, dx \\
= 2\pi \int_0^1 (x^{3/2} - x^3) \, dx \\
= 2\pi \left[ \frac{2}{5} x^{5/2} - \frac{x^4}{4} \right]_0^1 \\
= 3\pi/10
\]

(2) Let $h$ be the depth into the tank. Then at a given $h$ the infinitesimal volume of a slice is $\pi (4 - h^2) \Delta h$. The distance to the rim is $h$. Hence
\[
\text{Work} = \int_0^2 \pi (4 - h^2) h \, dh \\
= \pi \left[ 2h^2 - h^4/4 \right]_0^2 \\
= 4\pi
\]

(3) The intersection points occur when $3 - x^2 = x^2 + 1$, or $x = \pm 1$. Hence, the area is given by
\[
\text{Area} = \int_{-1}^1 (3 - x^2 - (x^2 + 1)) \, dx = \int_{-1}^1 (2 - 2x^2) \, dx \\
= (2x - 2x^3/3) \bigg|_{-1}^1 = 8/3
\]

For the center of mass, notice that the region is symmetric with respect to $x \mapsto -x$. Hence, $\bar{x} = 0$. For $\bar{y}$, we need to calculate
\[
M_x = \frac{1}{2} \int_{-1}^1 ((3 - x^2)^2 - (x^2 + 1)^2) \, dx \\
= \frac{1}{2} \int_{-1}^1 (8 - 8x^2) \, dx \\
= 4(x - x^3/3) \bigg|_{-1}^1 = 16/3
\]

Hence, $\bar{y} = (16/3)/(8/3) = 2$. Alternatively, notice that the region is symmetric about the axis $y = 2$! Indeed, $f(x) - 2 = 1 - x^2 = -(x^2 - 1) = -(g(x) - 2)$.

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(4) We have \( \dot{x} = \sin t \), \( \dot{y} = 1 - \cos t \), so

\[
\dot{x}^2 + \dot{y}^2 = \sin^2 t + (1 - \cos t)^2 = 2 - 2 \cos t = 4 \sin^2 (t/2)
\]

Hence, the length is

\[
\int_0^\pi \sqrt{\dot{x}^2 + \dot{y}^2} \, dt = \int_0^\pi 2 \sin (t/2) = -4 \cos (t/2) \bigg|_0^\pi = 4
\]