(1) (a) Since $na_n \to 2$, we may assume $1 \leq na_n \leq 3$, or $1/n \leq a_n$ and $a_n^2 \leq 9/n^2$. The first inequality implies that $\sum a_n$ diverges (comparison with the harmonic series) and the second inequality implies $\sum a_n^2$ converges (comparison with the $p$-series, $p = 2$). (b) This is a geometric series with $a = 1/3$ and $r = -2/3$. Hence, the sum is

$$\frac{a}{1 - r} = \frac{1/3}{1 + 2/3} = \frac{1}{5}$$

(c) This is an alternating series with

$$a_n = \frac{n + 1}{n^2 + n + 1}$$

Since $a_n \to 0$ (by L'Hôpital's rule), it suffices to show $a_n \geq a_{n+1}$ for $n$ sufficiently large. This follows directly by algebra, or notice that $f'(x) < 0$ (and so $f$ is decreasing) for $x$ large, where $f(x) = (x + 1)/(x^2 + x + 1)$.

(2) (a) By the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{(n + 1)^{2n+2} (2n)!}{(2n + 2)! n^{2n}} = \frac{(1 + 1/n)^{2n+2}}{(2 + 1/n)(2 + 2/n)^n} \to \frac{e^2}{4}$$

So the radius of convergence is $4/e^2$. (b) By the root test,

$$\sqrt[n]{|a_n|} = \frac{(\ln n)^{3/n}}{n^{2/n}} \to 1$$

(note that since $\ln(n)/n \to 0$ and $\ln(\ln n)/n \to 0$ by l'Hôpital's rule, we have $n^{1/n} = \exp(\ln n)/n \to 1$ and $(\ln n)^{1/n} = \exp(\ln(\ln n)/n) \to 1$). So the radius of convergence in this case is 1.

(3) For (a),

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$$

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For (b),

\[
\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \\
\frac{1}{2 - t^2} = \frac{1}{2} \frac{1}{(1 - t^2/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n} \\
\int_0^x \frac{dt}{2 - t^2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n (2n + 1)}
\]

(4) The \(n\)-th coefficient in the expansion is \(f^{(n)}(0)/n!\). So

\[
\frac{f^{(10)}}{10!} = \frac{-1}{4^5 (5!)^2} \quad f^{(10)}(0) = -\frac{10!}{4^5 (5!)^2}
\]