Asymptotics of determinants from functional integration

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The expression for the determinant of the Laplace operator is used in terms of functional integration to compute the asymptotic behavior on degenerating Riemann surfaces. In the case of the Arakelov metric, the information is sufficiently precise to give a value for the absolute constants appearing in bosonization formulas.

I. INTRODUCTION

The behavior of determinants of Laplace operators on degenerating Riemann surfaces is of particular interest in string theories, where analytic information about the string integrand near the boundary of moduli space is essentially strong enough to determine the integrand uniquely. For the case of the hyperbolic metric on surfaces of genus of at least 2, the determinant may be expressed in terms of the lengths of closed geodesics. These serve as real parameters of the degeneration, and an analysis of this expression gives the asymptotic behavior. Alternatively, one may look at Green's functions and exploit the "bosonization" formulas for determinants. This gives especially precise information in the case of the Arakelov metric. In this paper, we would like to show how the expression for the determinant in terms of functional integrals readily yields the asymptotics of the determinant and the next order term. This is a development of ideas presented in Refs. 1 and 6.

Our main results are as follows: For a compact Riemann surface $M$ with metric $g$, let

$$Z_g = \left[ \frac{8\pi^2 \det' \Delta_g}{\text{Area}(M,g)} \right]^{-1/2},$$

where the determinant of the Laplace operator is given by zeta regularization. We construct a family $M, \xi$ of surfaces degenerating as $t \to 0$ to a surface with a node. There are two possibilities: if the node separates the singular surface, then the two components $M, \xi$ and $M, \xi$ are compact Riemann surfaces, and the node is the identification of punctures on $M, \xi$ and $M, \xi$. Let $g, \xi$ and $g, \xi$ be metrics on the components, and $g, \xi$ a family of metrics on $M, \xi$. Then we show that

$$\frac{Z_{g, \xi}}{Z_{g, \xi}} = \left( \frac{\det_D \Delta_0(D, \xi)}{\det_D \Delta_0(A, \xi)} \right) \times \exp \left\{ \frac{1}{2} \left( -S_L(\sigma, R, D, \xi) - S_L(\sigma, R, D, \xi) + S_L(\sigma, D, A, \xi) - S_L(\sigma, A, \xi) \right) \right\},$$

where $g$ is a metric on the compact surface $M$. The difference in the power of $\log(\epsilon/|t|^{1/2})$ is related to the creation of an extra zero mode in the case of degeneration to a separating node.

In Sec. III, we consider the example of the Arakelov metric. Combining the above formulas with our previous results, we are able to determine the constant relating the Faltings invariant $\delta(M)$ to the determinant.

$$\delta(M) = c_n - 6 \log \left[ \frac{\det' \Delta_g}{\text{Area}(M,g)} \right],$$

where $g$ is the Arakelov metric, and

$$c_n = (-1) c_0 + c_1,$$

$$c_0 = -24 c_0 (-1) - 6 \log 2\pi - 2 \log 2 - 5,$$

$$c_1 = -8 \log 2\pi.$$

Finally, we note that the constant $c_n$ is related to the absolute constants appearing in the bosonization formulas (see Refs. 9-11).

II. DERIVATION

In this section, we use the functional integral formula for the determinant of the Laplace operator to obtain an expression from which the asymptotics of the determinant on degenerating Riemann surfaces are easily obtained.

A. Degeneration to a separating node

Fix two compact Riemann surfaces $M, \xi$, of genus $h, > 0$. Choose points $p, \xi$ (henceforth both denoted simply by $p$) and local coordinates $z, \xi$ centered at $p$. When $M, \xi$ and $M, \xi$ are endowed with metrics, we will always assume the coordinates to be normalized. By this we mean that if the metric is expressed in the coordinates $z, \xi$,

$$d^2 s = 2g_{z, \xi} dz, dz, \xi,$$

then $g_{z, \xi}(0) = 1$.

The degenerating family $M, \xi$ is constructed as follows: From $M, \xi$ and $M, \xi$, remove the disks $|z| < |t|$ and identify the annuli $|t| < |z| < 1$ by the equation, $z, z, \xi = t, M, \xi$, is then an
analytic family of Riemann surfaces, compact of genus \( h = h_1 + h_2 \) for \( t \neq 0 \), and “stable” in the sense of Deligne-Mumford over the fiber \( t = 0 \) (see Ref. 12).

D. Degeneration to a nonseparating node

Fix a compact Riemann surface \( M \) of genus \( h \). Choose two points \( a, b \) and local coordinates \( z_a \) and \( z_b \), respectively. As above, we construct a family \( M_t \) by cutting out the disks \( |z_a| < \frac{1}{4} \) and \( |z_b| < \frac{1}{4} \) and identifying the remaining regions \( |z| < |z_a| < 1 \) by \( z_a, z_b = t \). For \( t \neq 0 \), \( M_t \) is then compact of genus \( h + 1 \). \( M_0 \) is the surface \( M \) with the “punctures” \( a, b \) identified. Notice that in this case, the node does not separate the surface.

C. Definition of determinants from functional integration

Given a metric \( g \) on \( M \) compatible with the complex structure, we define the \( L^2 \) norm on real-valued functions \( \phi \) on \( M \) with respect to \( g \),

\[
\|\phi\|^2 = \int_M d^2 z \sqrt{|g|} |\phi|^2.
\]

We then fix a path integral measure \( D\phi \) by,

\[
1 = \int D\phi \ e^{-\frac{1}{8\pi} \|\phi\|^2}.
\]

We will denote by \( \phi^* \) the projection onto the orthogonal complement of the constant functions. Hence,

\[
\phi^* = \phi - \frac{1}{\text{Area}(M,g)} \int_M d^2 z \sqrt{|g|} \phi.
\]

We can now derive the normalization of \( D\phi^* \) from (2.1). Let \( A = \text{Area}(M,g) \). Then,

\[
1 = \int D\phi^* \ e^{-\frac{1}{8\pi} \|\phi^*\|^2} = \frac{8\pi^2}{A} \int D\phi^* \ e^{-\frac{1}{8\pi} \|\phi^*\|^2},
\]

or

\[
\int D\phi^* \ e^{-\frac{1}{8\pi} \|\phi^*\|^2} = \left( \frac{8\pi^2}{A} \right)^{1/2}.
\]

Choosing zeta function regularization, a similar argument shows

\[
Z_{\phi} = \int D\phi^* \ e^{-I(\phi^*)} = \left[ \frac{8\pi^2 \det \Delta_g}{\text{Area}(M,g)} \right]^{-1/2},
\]

where the action \( I \) is defined by

\[
I(\phi) = \frac{1}{8\pi} \int_M d^2 z \sqrt{|g|} |\nabla \phi|^2.
\]

D. Factorization of the path integral—Separating node case

We now localize the right-hand side of (2.3). Recall the construction of Sec. II A. For small \( \epsilon \), \( |t| < \epsilon < 1 \), we define three curves:

\[
C_1 = \{ |z_1| = \epsilon \},
\]

\[
C_2 = \{ |z_2| = \epsilon \},
\]

\[
C_3 = \{ |z_1| = |z_2| = |t|^{1/2} \}.
\]

Furthermore, let \( A'_1 \) be the annulus bound by the curve \( \{ |z| = |t|^{1/2} \} \) and \( C'_1 \), and let \( D'_t \) be the disk in \( M_1 \) bound by the curve \( C'_1 \).

By the “sewing” property of functional integration, (2.3) becomes

\[
\int D\phi^* \ e^{-I(\phi^*)} = \int D\phi_1 D\phi_2 D\psi_1 D\psi_2 D\beta_1 D\beta_2 \times \sum D\alpha^* \times e^{-I(\phi_1) - I(\phi_2) - I(\psi_1) - I(\psi_2)},
\]

(2.5)

where \( \phi_1 \) maps \( R'_1 = M_1 - D'_t \) to the reals with boundary values

\[
\phi_1 |_{\partial D'_t} = \beta_1.
\]

Likewise for \( \phi_2, \psi_1 \) maps \( A'_1 \) to \( R \) with boundary values \( \beta_2 \) and \( \alpha^* \), respectively. Likewise for \( \psi_2 \). While no satisfactory proof of the “sewing” property (2.4.1) for functional integrals on Riemann surfaces exists, specific examples can be verified directly. In the Appendix, we show that (2.5) holds for the case of a disk cut out from a sphere.

Following D’Hoker-Phong, we decompose the \( \phi \)’s and \( \psi \)’s into two pieces,

\[
\phi_i = \tilde{\phi}_i + \chi_i, \quad \psi_i = \tilde{\psi}_i + \xi_i, \quad i = 1,2.
\]

where \( \tilde{\phi}_i \) and \( \tilde{\psi}_i \) have zero boundary conditions, and \( \chi_i \) and \( \xi_i \) are harmonic. Note that, for example,

\[
I(\phi_1) = I(\tilde{\phi}_1) + \frac{1}{8\pi} \int_{\partial D'_t} dnu \nabla \tilde{\phi}_1 \partial_\nu \tilde{\phi}_1,
\]

where \( n^\mu \) is the outward normal. Using the above, and by translation invariance of the measures, (2.5) becomes

\[
Z_{\phi} = \int D\tilde{\phi}_1 D\tilde{\phi}_2 D\tilde{\psi}_1 D\tilde{\psi}_2 e^{-I(\tilde{\phi}_1) - I(\tilde{\phi}_2) - I(\tilde{\psi}_1) - I(\tilde{\psi}_2)}
\]

\[
\times \sum D\beta_1 D\beta_2 D\alpha^* \exp \left\{ -\frac{1}{8\pi} \int_{\partial D'_t} dnu \nabla \beta_1 \partial_\nu \beta_1 - \frac{1}{8\pi} \int_{\partial A'_1} dnu \nabla \xi_1 \partial_\nu \xi_1 \right\}.
\]

The integrals in the top line just give the regularized determinants of the Dirichlet problem for the regions \( R'_1 \) and \( A'_1 \). We expand the boundary values in Fourier series,

\[
\beta_j = \sum_{m} b_j^m e^{im\phi},
\]

and define \( \beta_j^* = \beta_j - b_j^0 \). Then the above expression becomes

\[
\sum D\alpha^* \exp \left\{ -\frac{1}{8\pi} \int_{\partial D'_t} dnu \nabla \beta_j \partial_\nu \beta_j - \frac{1}{8\pi} \int_{\partial A'_1} dnu \nabla \xi_j \partial_\nu \xi_j \right\}.
\]
\[ Z_g = \left[ \det_D \Delta_x (R^1) \det_D \Delta_x (R^2) \det_D \Delta_x (A^1) \det_D \Delta_x (A^2) \right]^{-1/2} \int \mathcal{D} \beta^* \mathcal{D} \beta \exp \left\{ - \frac{1}{8\pi} \int_{\partial D^i} d^{n_{D^i}} \xi, \partial_x \xi, - \frac{1}{8\pi} \int_{\partial A^i} d^{n_{A^i}} \xi, \partial_x \xi \right\}. \tag{2.6} \]

Now consider the compact surfaces \( M_j \) with metrics \( g_j \). A derivation similar to the one above shows

\[ Z_{g_j} = \left[ \det_D \Delta_x (R^1) \det_D \Delta_x (D^1) \det_D \Delta_x (A^1) \det_D \Delta_x (A^2) \right]^{-1/2} \int \mathcal{D} \beta^* \mathcal{D} \beta \exp \left\{ - \frac{1}{8\pi} \int_{\partial D^i} d^{n_{D^i}} \xi, \partial_x \xi, - \frac{1}{8\pi} \int_{\partial A^i} d^{n_{A^i}} \xi, \partial_x \xi \right\}, \]

where \( \xi_j \) is harmonic on \( D^i_j \) and has boundary values \( \beta_j \). Combining this with (2.6), we have

\[ \frac{Z_{g_i}}{Z_{g_k}} = \left[ \det_D \Delta_x (R^1) \det_D \Delta_x (R^2) \det_D \Delta_x (A^1) \det_D \Delta_x (A^2) \right]^{-1/2} \int \mathcal{D} \beta^* \mathcal{D} \beta \exp \left\{ - \frac{1}{8\pi} \int_{\partial D^i} d^{n_{D^i}} \xi, \partial_x \xi, - \frac{1}{8\pi} \int_{\partial A^i} d^{n_{A^i}} \xi, \partial_x \xi \right\}. \tag{2.7} \]

This expression is actually quite tractable, despite its appearance. The integrals (we shall call them harmonic integrals) may be evaluated explicitly. The determinants on the \( R^i \)'s can be related through the Liouville action. The remaining determinants are evaluated on disks or annuli, and these have closed expressions. We now proceed to discuss all these quantities in detail.

**E. Evaluation of harmonic integrals**

In this section, we compute

\[ \int \mathcal{D} \beta \mathcal{D} \beta \exp \left\{ - \frac{1}{8\pi} \int_{\partial D^i} d^{n_{D^i}} \xi, \partial_x \xi, - \frac{1}{8\pi} \int_{\partial A^i} d^{n_{A^i}} \xi, \partial_x \xi \right\}. \tag{2.8} \]

Such integrals have been discussed in Ref. 13. We consider the general situation of an annulus \( A \) with inner radius \( r_1 \) and outer radius \( r_2 \) and with boundary conditions \( \alpha \) and \( \beta \), respectively. Let \( \chi \) be the unique harmonic function on \( A \) with the given boundary conditions. If we expand \( \alpha \), \( \beta \), and \( \chi \) into Fourier coefficients,

\[ \alpha = \sum_{n \in \mathbb{Z}} a_n e^{i n \theta}, \quad \beta = \sum_{n \in \mathbb{Z}} b_n e^{i n \theta}, \quad \chi = \sum_{n \in \mathbb{Z}} c_n(r) e^{i n \theta}, \]

we find the solutions,

\[ c_0(r) = A_0 + B_0 \log r, \quad c_n(r) = A_n r^n - B_n r^{-n}, \quad n \neq 0. \]

Imposing the boundary conditions, we have,

\[ A_0 = \frac{a_0 \log r_2 - b_0 \log r_1}{\log r_2 / r_1}, \quad A_n = \frac{a_n r_2^n - b_n r_1^n}{r_2^n - r_1^n}, \]

\[ B_0 = \frac{b_0 - a_0}{\log r_2 / r_1}, \quad B_n = \frac{b_n r_2^n - a_n r_1^n}{r_2^n - r_1^n}, \quad n \neq 0. \]

By computation,

\[ \int_{\partial A} d^{n_{A}} \chi, \partial_x \chi = 2\pi (b_0 - a_0)^2 + 4\pi \sum_{n=1}^{\infty} n b_n b_{-n} \lambda_n + 4\pi \sum_{n=1}^{\infty} n a_n a_{-n} \lambda_n - 16\pi \sum_{n=1}^{\infty} n b_n b_{-n} \frac{r_2^n + r_1^n}{r_2^n - r_1^n}, \tag{2.9} \]

where, \( \lambda_n = (r_2^n + r_1^n)/(r_2^n - r_1^n) \).

On the space of functions: \( \alpha = \sum_{n=1}^{\infty} a_n e^{i n \theta}, \quad \beta = \sum_{n=1}^{\infty} b_n e^{i n \theta} \), we have the inner product:

\[ \langle \alpha, \beta \rangle = (1/2\pi) a_0 \beta_0 + (\alpha^* \beta) + (\beta^* \alpha^*), \]

where, \( (\alpha^* \beta) = \sum_{n=1}^{\infty} n a_n b_{-n} \). The path integrals are normalized such that \( 1 = \int \mathcal{D} \alpha^* e^{-\frac{1}{2} \langle \alpha, \alpha^* \rangle} \). Computing as in Sec. II C, we find

\[ \int \mathcal{D} \alpha^* e^{-\frac{1}{2} \langle \alpha, \alpha^* \rangle} = (2\pi)^{-1}. \tag{2.10} \]

Applying the result (2.9) to the integral (2.8), we have,
The operators $Q - 1$ and $T$ are trace class, depending on $r_1$ and $r_2$, and,

$$
\frac{Q - 1}{T - 0} \quad \text{as } r_1 \to 0.
$$

We may therefore take the limit. Using (2.10),

$$
\int \mathcal{D} \alpha^* e^{iQ_1 \alpha}, \quad (Q_2 = 2\pi r^2, \quad (Q r^2)|_{1/2}, \quad (Q r^2)|_{1/2})
$$

$$
\to (2\pi)^{-1} \exp \left\{ \frac{1}{2} (\beta_1 \beta_1) \right\},
$$

as $r_1 \to 0$.

A similar computation is easily carried out for harmonic integrals over the disk of radius $r_2$ with boundary values $\beta_2$.

We find,

$$
\int_{D^2} d
\Xi \Xi \partial \partial \Xi = 4\pi (\beta_1 \beta_1).
$$

Putting these results together, we have for the quotient of harmonic integrals in (2.7),

$$
\frac{\mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1}{\mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1 \mathcal{D} \beta_1} = 2 \log \frac{e}{|t|^{1/2}} H_i(t, e).
$$

(2.11)

$H_i$ is continuous, and $\lim_{r \to 0} H_i(t, e) = 1$ for fixed $e$.

F. Case of a nonseparating node

In (2.11) the appearance of the $\log(e/|t|^{1/2})$ was due to the integration over both $b_0^1$ and $b_0^2$. This corresponds roughly to the creation of an extra zero mode as the surface $M$ is pulled apart. We shall see that the situation is quite different for the case where $M$ degenerates to a surface with a nonseparating node.

Recall the construction of Sec. II B. Again, we choose $\epsilon$ small and define three curves:

- $C_1^* = \{|z_1| = \epsilon\}$
- $C_2^* = \{|z_2| = \epsilon\}$
- $C_3^* = \{|z_3| = \epsilon\}$

Then we have

$$
\int d\phi d\psi = e^{-i(\phi - i\psi)} - e^{-i(\phi + i\psi)},
$$

where $\phi$, $\psi_\alpha$, and $\psi_\beta$ are defined by analogy with Sec. II D. We split the functions into harmonic pieces and functions with zero boundary values,

$$
\phi = \phi + \chi, \quad \psi_\alpha = \psi_\alpha + \zeta_\alpha, \quad \psi_\beta = \psi_\beta + \zeta_\beta,
$$

and simplify,

$$
Z \equiv \left[ \frac{\det_d \Delta_1 (R_4) \det_d \Delta_1 (A_4^*) \det_d \Delta_2 (A_4^*)}{\det_d \Delta_1 (R_4) \det_d \Delta_1 (A_4^*) \det_d \Delta_2 (A_4^*)} \right]^{1/2}
$$

$$
\times \frac{\mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta}{\mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta \mathcal{D} \alpha \mathcal{D} \beta} = 2 \pi \int d\alpha \exp \left\{ - \frac{1}{8\pi} \left\{ \int_{\partial b_1^1} d\alpha_1 \partial \partial \alpha_1 + \int_{\partial b_1^2} d\alpha_2 \partial \partial \alpha_2 + \int_{\partial b_2} d\alpha_3 \partial \partial \alpha_3 \right\} \right\}.
$$

(2.12)

Where $\zeta_\alpha$ is harmonic on $D_4^*$ and has boundary values $\mu_\alpha$, etc. Referring to the result (2.9), and expanding $\mu$ by its Fourier coefficients

$$
\mu_\alpha = \sum_{k \in \mathbb{Z}} m_k e^{i\alpha_k},
$$

we have,

$$
\int_{\partial b_1^1} d\alpha_1 \partial \partial \alpha_1 = 2\pi \left( m_0 - a_0 \right)^2 + 4\pi (Q\mu_\alpha^* \tilde{\mu}_\alpha^*) + 4\pi (Q\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*) + 8\pi (T\alpha_\alpha^* \tilde{\mu}_\alpha^*)
$$

Thus we may evaluate the harmonic integral

$$
\int d\alpha \exp \left\{ - \frac{1}{8\pi} \left\{ \int_{\partial b_1^1} d\alpha_1 \partial \partial \alpha_1 + \int_{\partial b_1^2} d\alpha_2 \partial \partial \alpha_2 + \int_{\partial b_2} d\alpha_3 \partial \partial \alpha_3 \right\} \right\} = \int d\alpha \exp \left\{ - \frac{1}{4\pi} \left\{ \int_{\partial b_1^1} d\alpha_1 \partial \partial \alpha_1 + \int_{\partial b_1^2} d\alpha_2 \partial \partial \alpha_2 + \int_{\partial b_2} d\alpha_3 \partial \partial \alpha_3 \right\} \right\}.
$$

(2.13)
with \( \lim_{t \to 0} H(t, \epsilon) = 1 \) for fixed \( \epsilon \). The fact that \( \log|t| \) is raised to the power 0.5 rather than 1 will mean that the singularity for the logarithm of the determinant carries an extra \( \log(-\log|t|) \) term [see Eq. (2.17)].

**G. The Liouville action**

We now recall the relationship between determinants on surfaces with boundary for conformally related metrics. Suppose that on a surface \( R \) with boundary we have metrics \( g \) and \( \hat{g} \) related by a conformal factor, \( g = e^{2\varphi} \hat{g} \). Then for the Dirichlet problem on \( R \), we have,

\[
\det_D \Delta g(R) = \det_D \Delta h(R) e^{S_e(u, R)}
\]

where

\[
S_e = \frac{1}{2} \int_R d\xi \sqrt{\hat{g}} \frac{\partial}{\partial \varphi} \partial_\varphi \partial_\varphi 
\]

Using this expression and (2.11), we can rewrite (2.7) as,

\[
\frac{Z_{h'}}{Z_{h}} = \left( \frac{\det_D \Delta_h(D_0)}{\det_D \Delta_0(A_0)} \right) 2 \log \left( \frac{\epsilon}{|t|^{1/2}} \right) H(t, \epsilon) 
\]

\[
\times \exp \left\{ - S_L(\sigma, R') - S_L(\sigma, R_2') 
\right. 
\]

\[
+ S_L(\sigma, D') + S_L(\sigma, D_2') - S_L(\sigma, A_2') 
\left. - S_L(\sigma, A_2') \right\} 
\]

(3.15)

Here, \( \sigma \) is the conformal factor relating \( g \) to \( g_0 \) on \( R \), and that relating \( g \) to the Euclidean metric in the annuli \( A \). \( \sigma \) is the conformal factor relating \( g_0 \) to the Euclidean metric on \( D_0 \). \( \Delta_0 \) is the Laplace operator in the Euclidean metric.

The form of (3.15) makes it easy to compute the asymptotics of the determinant when one has sufficient estimates for the behavior of the metric. The Euclidean determinants, as mentioned, may be evaluated explicitly—we recall these results for reference (see Ref. 14):

\[
\det_D \Delta_0(D_0) = 2 - \frac{1}{n} \log -\frac{1}{2} \log|t|^{1/2} 
\]

\[
\times \exp \left\{ - 2\xi^{(1)}(-1) - 5/12 \right\} 
\]

\[
\det_D \Delta_0(A_0) = e^{-1/2 - 1/3} 
\]

\[
\times \log \left( \epsilon/|t|^{1/2} \right) \left( f(|t|/\epsilon^2) \right) - 2, 
\]

(2.16)

where \( \xi(s) \) is the Riemann zeta function, and \( f \) is the partition function.

\[
f(x) = \prod_{n=1}^{\infty} \left( 1 - x^n \right)^{-1}.
\]

For the case of a nonseparating node, we have, from (2.12) and (2.13),

\[
Z_{h'} = \left( \frac{\det_D \Delta_0(D_0)}{\det_D \Delta_0(A_0)} \right) \left( 2\pi \right)^{1/2} \left[ \log \frac{\epsilon}{|t|^{1/2}} \right]^{1/2} H(t, \epsilon) 
\]

\[
\times \exp \left\{ - S_L(\sigma, R') + S_L(\sigma, D') + S_L(\sigma, D_2') 
\right. 
\]

\[
- S_L(\sigma, A_2') \right\}, 
\]

(2.17)

with the obvious definitions.

At this point, we can see the difference in the behavior of the determinant depending upon whether the degeneration is to a separating or nonseparating node. Roughly speaking, one expects that for "reasonable" families of metrics \( g \), the Liouville actions will contribute singularities of the order \( -\log|t| \). Comparing (2.15), (2.16), and (2.17), we see that asymptotically \( \log \det \Delta \) has an additional term \( \log(-\log|t|) \) if and only if the node is nonseparating.

**III. EXAMPLE—THE ARAKELOV METRIC**

We use the results of Sec. II to obtain the asymptotics of the determinant in the Arakelov metric. Comparing this with our previous results on the asymptotics of the Faltings invariant, we are able to obtain the exact additive constant relating the two.

**A. Estimating the Liouville action**

We now evaluate the terms in the exponential of (2.15) for surfaces degenerating with the Arakelov metric.\(^7\) We shall treat only the case of degeneration to a separating node—recall the construction of Sec. II A. The asymptotics of the metric are\(^9\)

\[
\log g_{a} = 2(h'2/h)^2 \log |t| + \log g_{a}^{(i)} 
\]

\[
- 4(h'2/h) \log G^{(i)}(z,w) + o(1),
\]

for \( z, w \in M \) and \( G^{(i)}(z,w) \) the Arakelov–Green's function for \( M \). A similar expression holds for \( z, v \).

Recall that in (2.15), the conformal factor near \( p \) relates the above metric to the Euclidean metric, and away from \( p \) the relation is with respect to \( g^{(i)} \). To simplify notation, we write \( g \) for \( g^{(i)} \) and \( \hat{g} \) for the Euclidean metric.

For \( z \) near the node \( p \), we use \( G^{(i)}(z,p) \sim |z| \), in local coordinates.

\[
\sigma(z) = (h'2/h) \log |z|^2 + \cdots, 
\]

(3.1)

\[
\partial_z \sigma(z) = - (h'2/h) \log |z| + \cdots. 
\]

(3.2)

Away from \( p \), we have,

\[
\sigma(z) = (h'2/h) \log |z| - 2(h'2/h) \log G^{(i)}(z,p) + \cdots, 
\]

(3.3)

\[
\partial_z \sigma(z) = - 2i(h'2/h) \mu^{(i)} + \cdots, 
\]

(3.4)

where \( \mu^{(i)} \) is the canonical metric on the surface \( M \).

Near \( p \) we compute, using (3.1) and (3.2), to order \( \epsilon \),

\[
\sigma' \partial_{\sigma} f_{\sigma} = \int_{|t|^{1/2}}^{r} d\rho \left( \frac{h'2}{h} \right)^{1/2} 
\]

\[
\frac{1}{r^2} 
\]

\[
\frac{1}{12\pi} \int_A \frac{d^2\xi}{\sqrt{\det \partial_\varphi \partial_\varphi}} 
\]

\[
= \frac{1}{3\pi} \int_0^{2\pi} d\theta \int_{|t|^{1/2}}^r d\rho \left( \frac{h'2}{h} \right)^{1/2} 
\]

\[
\frac{1}{r^2} 
\]

\[
\]

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\[
\frac{1}{3} \left( \frac{h_2}{h} \right)^2 \log |t| - \frac{2}{3} \left( \frac{h_2}{h} \right)^2 \log \epsilon,
\]
\[= - \frac{1}{6\pi} \int_{\partial \Sigma} \! \! \! \! \! \! ds \, k_2 \sigma = - \frac{1}{6\pi} \int_{\partial \Sigma} \! \! \! \! \! \! d\theta \{ \sigma(\epsilon) - \sigma(|t|^{1/2}) \},
\]
\[= \frac{2}{3} \frac{h_2}{h} \log \epsilon - \frac{1}{3} \frac{h_2}{h} \log |t|.
\]

The other terms in the Liouville action are order \( \epsilon \). Away from \( p \), we use (3.3) and the expression for the curvature of the Arakelov metric \( \partial_2 \partial_2 \log g^2 = 4\pi i(h - 1) \mu_{d_2}: \)

\[- \frac{1}{6\pi} \int_{R^*_1} dv \, K\sigma = \frac{2}{3} (h_1 - 1) \int_{R^*_1} \mu_1 \left( \frac{h_2}{h} \right)^2 \log |t|.
\]

\[- \frac{2}{3} \frac{h_2}{h} \log G^{(1)}(z,p).
\]

The computations for \( R^*_2 \) are identical. Adding these together, we have

\[- \frac{1}{12\pi} \int_{R^*_1} d^2 \Omega \sqrt{g_{\partial_\sigma} \partial_\sigma \partial_\sigma} = - \frac{1}{12\pi} \frac{4}{3} \left( \frac{h_2}{h} \right)^2 \int_{R^*_1} dv(z) g^{\partial_\sigma} \partial_\sigma \log G^{(1)}(z,p) \partial_\sigma \log G^{(1)}(z,p)
\]

\[- \frac{1}{3\pi} \left( \frac{h_2}{h} \right)^2 \int_{R^*_1} dv(z) \log G^{(1)}(z,p) \Delta_\epsilon \log G^{(1)}(z,p)
\]

\[- \frac{1}{3\pi} \left( \frac{h_2}{h} \right)^2 \int_{\partial R^*_1} d\eta^\mu \log G^{(1)}(z,p) \partial^\mu \log G^{(1)}(z,p).
\]

The first term vanishes as \( \epsilon \to 0 \) by \( \partial_2 \partial_2 \log G(z,w) = \pi i \mu_{d_2} \), \( z \neq w \), and the above normalization. The second term is

\[- \frac{1}{3\pi} \left( \frac{h_2}{h} \right)^2 \int_0^{2\pi} \epsilon \, d\theta \frac{1}{\epsilon} \log G^{(1)}(z,p) \sim \frac{1}{3} \left( \frac{h_2}{h} \right)^2 \log \epsilon.
\]

The constants \( c_1 \) and \( c_2 \) are identical. Adding these together, we have

\[S_L(\sigma_1, R^*_2) + S_L(\sigma_2, R^*_1) + S_L(\sigma_1, A^*_2) + S_L(\sigma_2, A^*_1) = \frac{2}{3} \frac{h_1 h_2}{h} \log |t| - \frac{1}{3} \log |t| + o(\epsilon, t).
\]

Furthermore, it is easy to see that \( S_L(\sigma_1, A^*_1) \), and \( S_L(\sigma_2, A^*_2) \) are also \( o(\epsilon) \).

### B. Asymptotics of the determinant

Combining the results of (2.15), (2.16), and (3.5), it follows that for \( g \), the Arakelov metric,

\[\lim_{t \to 0} \frac{\det' \Delta_{s_1}}{\text{Area}(M, g_1)} = - \frac{2}{3} \frac{h_1 h_2}{h} \log |t|
\]

\[= \frac{\det' \Delta_{s_1}}{\text{Area}(M, g_1)} + \frac{\det' \Delta_{s_2}}{\text{Area}(M, g_2)} - \frac{c_0}{6},
\]

where \( c_0 = -24 \epsilon \log (-1) - 6 \log 2\pi - 2 \log 2 - 5 \).

The Faltings invariant is related to the \( \zeta \)-regularized determinant with respect to the Arakelov metric by 2,9

\[\delta(M) = c_0 - 6 \log \left[ \det' \Delta_{s} / \text{Area}(M,g) \right].
\]

### APPENDIX

Here we give an example of the "sewing" property (2.5) for functional integrals on Riemann surfaces. The glueing of the two ends of a cylinder to obtain a torus was shown in Ref. 6 and other examples in the plane can be found in Ref. 13.
We consider the example of gluing two disks to obtain a sphere.

Let \( M \) be the sphere of radius 1 (regarded as \( \mathbb{C} \cup \{ \infty \} \)) with constant curvature metric \( g \) given by

\[
d^2s = 4 \left( \frac{\left| dz \right|^2}{1 + |z|^2} \right).
\]

The conformal factor relating \( g \) to the Euclidean metric \( \hat{g} \), defined by \( g = e^{2\sigma} \hat{g} \), is

\[
\sigma = \log 2 - \log(1 + |z|^2).
\]

From (2.3) we have

\[
(2\pi \det' \Delta_{\hat{g}})^{-1/2} = \int D\phi^* e^{-I(\phi)}.
\]  

(A1)

In the local coordinate \( z \), we cut out a disk \( B(R) = \{ |z| < R \} \). We must evaluate \( \det_D \Delta_{\hat{g}}(B(R)) \) where \( D \) denotes the Dirichlet problem. From Sec. II G, this reduces to evaluating the Euclidean determinant and the Liouville action (2.14). This is easily done—the result is

\[
S_L(\sigma, R) = -\frac{1}{3} \log 2 + \frac{R^2}{1 + R^2} - \frac{1}{3(1 + R^2)} + \frac{1}{3}.
\]

Substituting \( R \) for \( \epsilon \) in (2.16), we have

\[
\det_D \Delta_{\hat{g}}(B(R)) = (2\pi)^{-1/2} R^{-1/3} \exp \left\{ -2\xi'(1) \right\}
\]

\[
\left( \frac{R^2}{1 + R^2} - \frac{1}{3(1 + R^2)} \right) \frac{1}{12}.
\]

(A2)

For the disk \( M - B(R) \), we use the coordinate \( 1/z \) to obtain

\[
\det_D \Delta_{\hat{g}}(B(1/R)) = (2\pi)^{-1/2} R^{-1/3} \exp \left\{ -2\xi'(1) \right\}
\]

\[
+ \frac{1}{1 + R^2} - \frac{R^2}{3(1 + R^2)} \frac{1}{12}.
\]

(A3)

We now decompose the path integral as in Sec. III D:

\[
\int D\phi^* e^{-I(\phi)} = \int D\phi_1 D\phi_2 D\alpha^* e^{-I(\phi_1) - I(\phi_2)}
\]

\[
= \left[ \det_D \Delta_{\hat{g}}(B(R)) \det_D \Delta_{\hat{g}}(B(1/R)) \right]^{1/2}
\]

\[
\times \int D\alpha^* \exp \left\{ -\frac{1}{8\pi} \int_{\partial B(R)} d\gamma^\mu \partial_\mu X_1 \right\}
\]

\[
- \frac{1}{8\pi} \int_{\partial B(1/R)} d\gamma^\mu X_2 \partial_\mu X_2 \right\}.
\]

The harmonic integral is evaluated as in Sec. II E. The answer is simply \((2\pi)^{-1}\). Putting this together with (A1), (A2), and (A3), we have for the determinant on the sphere, \( \det' \Delta_{\hat{g}} = \exp \left( -4\xi'(-1) + 1/2 \right) \), which is the result obtained directly by evaluating the zeta function for the eigenvalues of the sphere (see Ref. 16).