SPECTRAL CONVERGENCE ON DEGENERATING SURFACES

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1. Introduction. The study of the spectrum of the Laplace operator has produced an extensive literature. (See [Cha] and the references therein.) Of special interest to recent applications has been the behavior of spectra on two-dimensional surfaces with degenerating metrics; for example, the case of hyperbolic metrics on Riemann surfaces is already quite complicated. (See [Hj], [Ji], [W1], [W2].) In this paper we show that, for a wide variety of degenerating metrics which have, however, quite different behavior from that of the hyperbolic metric, the spectrum converges to the spectrum of the surface with the degenerate metric.

Specifically, we consider surfaces $M_0$ with a singular metric, where the singularity in local coordinates is quasi-isometrically a cone. (See Sect. 2 for our model.) Such singularities were studied first by Cheeger [Che1] and subsequently by various authors, particularly in the context of $\bar{\partial}$, Dirac, and other first-order operators. (See [Chou], [BS], [S1], [S2].) It is a fundamental fact about metrics with cone singularities that the Laplacian $\Delta_0$ on $M_0$ still has a discrete spectrum $\text{Spec}(\Delta_0) = \{\lambda_i(0)\}_{i=0}^\infty$, which we order $0 = \lambda_0(0) < \lambda_1(0) < \lambda_2(0) < \ldots$. The natural question which then arises is the following: suppose we are given compact surfaces $M_t$ with degenerating metrics $g_t$ converging as $t \to 0$ to a metric on $M_0$ which has a cone singularity $p$. The singularity is assumed to be a double point; that is, locally we have two cones joined at their vertices. The noncompact surface $M_0 \setminus \{p\}$ may or may not be connected, and we shall refer to these two possibilities as the nonseparating and separating cases, respectively. We are interested in when $\text{Spec}(\Delta_t) \{\lambda_i(t)\}_{i=0}^\infty$ converges to $\text{Spec}(\Delta_0)$. To state the results precisely, we fix some notation: let $\{\varphi_i(t)\}_{i=0}^\infty$ denote a complete orthonormal basis of eigenfunctions with eigenvalues $\lambda_i(t)$, and for $\lambda > 0$ define the kernel function

$$K_t(x, y; \lambda) = \sum_{\lambda_i(t) < \lambda} \varphi_i(t)(x)\varphi_i(t)(y).$$

By spectral convergence we mean the following:

(*) Spectral convergence

(i) For all $i \geq 1$, $\lim_{t \to 0} \lambda_i(t) = \lambda_i(0)$;

(ii) for any sequence $t_j \to 0$ there exists a subsequence $t'_j \to 0$ such that for all $i \geq 1$

$$\lim_{j \to \infty} \varphi_i(t'_j) = \varphi_i(0)$$

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uniformly on compact subsets of $M_0 \setminus \{p\}$ for some choice of complete orthonormal basis $\{q_i(0)\}$ of eigenfunctions for $M_0$;

(iii) for any $\lambda > 0$, $\lambda \notin \text{Spec}(\Delta_0)$,

$$\lim_{t \to 0} K_t(x, y; \lambda) = K_0(x, y; \lambda)$$

uniformly on compact subsets of $M_0 \setminus \{p\} \times M_0 \setminus \{p\}$.

In Section 2 we recall the definition of a cone metric and some basic results. We shall construct a model for $M_t$, $0 \leq t \leq 1$, degenerating as $t \to 0$ to a surface with a cone metric, and in the subsequent two sections we prove the following theorem.

**Theorem A.** For $M_t$ a family of compact Riemannian surfaces degenerating as $t \to 0$ to a surface with cone metric, we have spectral convergence ($\ast$).

Our main tool for the proof of Theorem A is the result of P. Li (Theorem 2.5 below) giving $L^\infty$ estimates on eigenfunctions in terms of the inverse squared of the isoperimetric constant. For our degeneration model, the constant, localized to the degenerating neighborhood, is bounded away from zero (Prop. 2.6), and Li's estimate may then be used to extract a converging subsequence of eigenfunctions. The theorem then follows by a min-max argument. All this occupies Sections 3 and 4.

In Section 5 we construct analytic families of compact Riemann surfaces $M_t$ of genus $g \geq 2$ (where $t$ is now in the unit disk $D \subset \mathbb{C}$) degenerating as $t \to 0$ to a surface $M_0$ with a node $p$. If $\mu_t$ denotes the Bergman metric on $M_t$ (see Def. 5.1), then $M_0$ has the metric $g/(\mu_0)$ on $M_t$, $i = 1, 2$ if $p$ is separating. If $p$ is nonseparating, then $M_0$ has the metric $(g - 1)/\mu$. However, in this case the elliptic tail becomes a "long, thin cylinder" as $t \to 0$. We prove the following theorem.

**Theorem B.** Let $M_t$ be a degenerating family of compact Riemann surfaces endowed with Bergman metrics $\mu_t$.

(i) If $M_0 \setminus \{p\}$ has two components, then as $t \to 0$, we have spectral convergence ($\ast$);

(ii) if $M_0 \setminus \{p\}$ is connected, then the set of limit points of $\text{Spec}(\Delta_t)$ as $t \to 0$ is dense in $[0, +\infty)$.

Finally, in Section 6 we study the admissible metrics of Arakelov $[A]$, normalized to have unit area. (See Def. 6.2.) In Proposition 6.6 we show that these metrics degenerate to "admissible cone metrics" which are supported on the component of $M_0 \setminus \{p\}$ with the larger genus. In the equal-genus separating case, we again have a long, thin cylinder.

**Theorem C.** Let $M_t$ be a degenerating family of compact Riemann surfaces with normalized admissible metrics.

(i) If $M_0 \setminus \{p\}$ has one component or has two components of unequal genus, then we have spectral convergence ($\ast$). (Spec($\Delta_0$) is the spectrum of the cone metric on the component of larger genus.)

(ii) If $M_t$ degenerates to two surfaces of equal genus, joined at a separating node, then the set of limit points of $\text{Spec}(\Delta_t)$ as $t \to 0$ is dense in $[0, +\infty)$. 

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2. Cone metrics and isoperimetric constants. In this section we recall the definition of a cone metric and its spectrum. We present a model for a surface with smooth metric degenerating to a cone metric. Finally, we introduce the isoperimetric constant $\mathcal{J}(C)$, and in Proposition 2.6 we show that, for the degeneration model, the constant is bounded away from zero.

Definition 2.1. Let $(N, \tilde{g})$ be a closed, smooth $(n-1)$-dimensional Riemannian manifold. The cone $C(N)$ on $N$ is defined as the space $(0, 1) \times N$ with metric

$$ds^2(r, x) = dr \otimes dr + r^2 \tilde{g}(x).$$

An $n$-dimensional manifold $M$ with metric $g$ defined on $M \setminus \{p\}$ is called a cone manifold, and $g$ is called a cone metric with conical singularity at $p$ if, for some choice of $N$ and some neighborhood $U$ of $p$, $U \setminus \{p\}$ is isometric to $C(N)$.

Of course, we may generalize this definition to include the case of several conical singularities. For simplicity, however, we shall always deal with one.

Let $g = g_{ij} dx_i \otimes dx_j$ have conical singularity at $p$. Then

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( \sqrt{\det g^{ij} \frac{\partial}{\partial x_j}} \right)$$

(where $g^{ij} = (g^{-1})^{ij}$) is a second-order differential operator acting on $C^0_0(M \setminus \{p\})$. We wish to extend $\Delta$ as an operator acting on the Hilbert space $L^2(M)$. For this we take the domain of $\Delta$ to consist of $L^2$ functions $f$ such that $|\nabla f|, \Delta f \in L^2(M)$. Since Stokes's theorem holds for cone manifolds ([Che2], Theorem 2.2), then by a theorem of Gaffney [G], the $L^2$-closure of $\Delta$ is selfadjoint. We call this closure the Laplacian of $M$ and continue to denote it by $\Delta$. Furthermore, we have the following theorem.

Theorem 2.2 ([Che1], Theorem 3.1).

(a) $\Delta$ acting on $L^2(M)$ has discrete spectrum, and each eigenvalue has finite multiplicity.

(b) An eigenfunction $\varphi$ of $\Delta$ with eigenvalue $\lambda$ is characterized by $\Delta \varphi - \lambda \varphi = 0$, with $\varphi, |\nabla \varphi| \in L^2(M)$. The eigenvalues may be ordered with multiplicity $0 = \lambda_0(M) \leq \lambda_1(M) \leq \cdots$.

In this paper we are interested in the case of two-dimensional manifolds and a slight generalization of the notion of conic singularity—namely, the case where $p$ is a double point. This may be regarded locally as the union of two cone surfaces with the singularity identified. It is natural to view such a singularity as arising from a pinched cylinder or annulus. Consider the following family $C_t$ of annuli with a metric: for $0 \leq t \leq 1$,

$$C_t = \{(x, y)| -1 \leq x \leq 1, 0 \leq y \leq 2\pi\}/\{(x, 0) \sim (x, 2\pi)\}$$
with metric $ds^2 = dx^2 + (t + (1 - t)x^2) dy^2$. From Definition 2.1 the metric on $C_0$ has a cone singularity with a double point as described above. In the following we shall refer to a family of compact, connected surfaces $M_t$ with Riemannian metrics $g_t$, $0 < t \leq 1$, as a conic degenerating family if $M_t$ contains a cylinder (henceforth referred to as the pinching annulus) which is uniformly quasi-isometric to $C_t$ and $g_t$ converges on $M_t \setminus C_t$ to a smooth Riemannian metric. In the limit we have a singular metric $g_0$ on $M_0$, quasi-isometric to one with a cone singularity at a double point $p$—notice that $M_0 \setminus \{p\}$ may or may not be connected. We refer to these two possibilities as the nonseparating and separating cases, respectively. We now turn to the isoperimetric constant.

**Definition 2.3.** For a compact Riemannian manifold $M^n$ of dimension $n$ without boundary, the Sobolev constant $\mathcal{S}(M)$ is defined to be supremum over all constants $c$ such that

$$\left( \int_M |\nabla f|^2 \right)^{\frac{n}{n-1}} \geq c \inf_{\alpha \in \mathbb{R}} \left( \int_M |f - \alpha|^{n/(n-1)} \right)^{n-1}$$

for all functions $f$ on $M$, and the isoperimetric constant is defined as

$$\mathcal{I}(M) = \inf_S \frac{\text{Area}(S)^n}{\min\{\text{Vol}(N_1), \text{Vol}(N_2)\}}^{n-1},$$

where $S$ ranges over all hypersurfaces in $M$ which divide $M$ into two components $N_1$ and $N_2$ with $\partial(N_1) = \partial(N_2) = S$.

On the other hand, for a compact Riemannian manifold $M^n$ with nonempty boundary $\partial M$, the Sobolev constant $\mathcal{S}(M)$ is defined as

$$\mathcal{S}(M) = \inf_f \frac{\left( \int_M |\nabla f|^2 \right)^{\frac{n}{n-1}}}{\int_M |f|^{n/(n-1)}}$$

where $f \neq 0 \in C_0^\infty(M \setminus \partial M)$, and the isoperimetric constant is defined as

$$\mathcal{I}(M) = \inf_D \frac{\{A(\partial D)\}^n}{\{V(D)\}^{n-1}}$$

where $D \subset M$ ranges over all open submanifolds of $M$ having smooth boundary satisfying $\overline{D} \cap \partial M = \emptyset$, and $A$ and $V$ denote the area and volume, respectively.

**Theorem 2.4.** (See [Cha], Theorems 4 and 12 in Chap. IV.) For any compact Riemannian manifold $M$ with boundary $\partial M$ ($\partial M$ may be empty),

$$\mathcal{I}(M) \leq \mathcal{S}(M).$$
If we treat the cone singularity as an interior point, then the definitions of $\mathcal{S}(M)$, $\mathcal{J}(M)$, and Theorem 2.4 easily generalize for surfaces $(n = 2)$ with cone singularities. For the proof of Theorems A, B, and C, we need the following $L^\infty$ bounds on eigenfunctions. (For simplicity we only state the case $n = 2$.)

**Theorem 2.5** (P. Li).

(i) If $M$ is a two-dimensional compact Riemannian manifold without boundary, then there is a constant $c$ independent of $M$ such that, for any eigenfunction $\varphi$ on $M$ with eigenvalue $\lambda \neq 0$,

$$\|\varphi\|_{L^\infty} \leq c \left( \frac{4\lambda}{\mathcal{J}(M)} \right)^2 V(M) \|\varphi\|_{L^2}.$$ 

(ii) If $M$ is compact with nonempty boundary, then for any eigenfunction $\varphi$ on $M$ of eigenvalue $\lambda$ with respect to Dirichlet boundary conditions, the same inequality holds for $\varphi$.

(iii) In either of the above cases, if we assume $M$ has a cone singularity, then the same inequality holds for eigenfunctions on $M$.

**Proof.** For parts (i) and (ii), see [Li] and [Cha, Sect. 4 in Chap. IV]. For part (iii) we note that Stokes's theorem holds for manifolds with cone singularities. (See [Che2].) Then the same proof works in this case as well.

We have the following uniform lower bound for $\mathcal{S}(C_t)$.

**Proposition 2.6.** There exists a constant $c > 0$ independent of $t$ such that for $0 < t \leq 1$

$$\mathcal{S}(C_t) \geq c > 0.$$ 

To estimate isoperimetric constants, we need the following theorem.

**Theorem 2.7** (F. Fiala, see [Fa]). Let $M$ be a Riemannian surface, $K$ its Gaussian curvature, and $K^+ = \max\{0, K\}$. Then for any simply connected domain $D$ in $M$,

$$L^2(\partial D) - 4\pi A(D) + 2\pi \int_D K^+ \geq 0$$

where $L(\partial D)$ is the length of the boundary $\partial D$ and $A(D)$ is the area of the domain $D$. In particular, if $K \leq 0$, then

$$L^2(\partial D) \geq 4\pi A(D).$$

**Lemma 2.8.** For $0 < t \leq 1$ the Gaussian curvature of $C_t$ is nonpositive.

**Proof.** This follows by direct computation.
Proof of Proposition 2.6. For \(1/2 < \alpha < 1\) the cones \(C_t\) form a compact family of compact surfaces. Thus, to bound the isoperimetric constants of \(C_t\) away from zero, it suffices to consider the case \(0 < \alpha < 1/2\). By the definition of isoperimetric constants, we need to estimate \(L^2(\partial D)/A(D)\) for all domains \(D \subset C_t\). According to a theorem of S.-T. Yau, however, it suffices to consider the situation where the domain \(D\) is connected. (See [Yau].) There are two cases to consider: (a) no component of \(\partial D\) is homotopic to a boundary component of \(C_t\); (b) at least one component of \(\partial D\) is homotopic to a boundary component of \(C_t\).

Case (a). In this case every component of \(\partial D\) is contractible in \(C_t\). Thus, we can assume that the domain \(D\) is simply connected. Otherwise, \(D\) may be embedded in the universal covering space \(\tilde{C}_t\) of \(C_t\), which is homeomorphic to \(\mathbb{R}^2\). Fill in the interior holes of \(D \subset \mathbb{R}^2\) and replace \(D\) by the newly filled one. In this way we increase the area of the domain, while decreasing the length of the boundary. Since the domain \(D\) is simply connected, by Theorem 2.7 and Lemma 2.8

\[
\frac{L^2(\partial D)}{A(D)} \geq 4\pi.
\]

Case (b). Since \(\partial D\) has at least one component homotopic to one component of the boundary of \(C_t\) and since \(D\) is connected, then \(\partial D\) has two components which are homotopic to the boundaries of \(C_t\), and all other components are contractible in \(C_t\). Filling in the holes bounded by the latter boundaries, we increase the area and decrease the length. Thus, we can assume that \(D\) is homeomorphic to a cylinder and that it has two boundaries, denoted by \(\gamma_1, \gamma_2\) (\(\gamma_1\) lies to the left of \(\gamma_2\)), which are homotopic to the boundaries of \(C_t\).

Step (i). First, we assume that \(\gamma_1\) and \(\gamma_2\) are rotationally symmetric, that is, for some \(-1 < \varepsilon_1 < \varepsilon_2 < 1\),

\[
D = \{(x, y) \in C_t | \varepsilon_1 < x < \varepsilon_2\}.
\]

It can be seen easily that it suffices to consider the case \(\varepsilon_1 = 0\), and \(0 < \varepsilon_2 = \varepsilon < 1\). Then

\[
L(\partial D) = 2\pi\{(t + (1 - t)\varepsilon)^{1/2} + t^{1/2}\},
\]

\[
A(D) = \int_0^{2\pi} dy \int_0^\varepsilon (t + (1 - t)x^2)^{1/2} dx.
\]

We are now going to estimate \(A(D_{\varepsilon,t})\) from above and \(L^2(\partial D_{\varepsilon,t})/A(D_{\varepsilon,t})\) from below. Depending on the relative size of \(t\) and \(\varepsilon\), there are two cases to consider.

First, we assume \(t \geq \varepsilon^2 > 0\). From the inequality \(\sqrt{1 + x} \leq 1 + x/2\) for \(x \geq 0\),
\[ A(D) = 2\pi \int_0^\varepsilon (t + (1 - t)x^2)^{1/2} \, dx \]

\[ \leq 2\pi \sqrt{t} \int_0^\varepsilon \left( 1 + \frac{1 - t}{2t} x^2 \right) \, dx \]

\[ = 2\pi \sqrt{t} \left( \varepsilon + \frac{1 - t \varepsilon^3}{3} \right) \]

\[ \leq 2\pi \left\{ t + \frac{1}{6} (1 - t)t \right\} \leq \frac{7}{3} \pi t, \]

\[ L^2(\partial D) \geq 4\pi^2 \{ t + (1 - t)e^2 \} \geq 8\pi^2 t. \]

Thus, for \( t \geq e^2 > 0, \)

\[ \frac{L^2(\partial D)}{A(D)} \geq \frac{8\pi^2 t}{\frac{7}{3} \pi t} = \frac{24}{7} \pi. \]

Next, assume \( 0 < t \leq e^2. \) Since \( \sqrt{1 + x} \leq 1 + \sqrt{x} \) for \( x \geq 0, \)

\[ A(D) = 2\pi \int_0^\varepsilon (t + (1 - t)x^2)^{1/2} \, dx \]

\[ \leq 2\pi \sqrt{t} \int_0^\varepsilon \left( 1 + \sqrt{\frac{1 - t}{t} x} \right) \, dx \]

\[ = 2\pi \sqrt{t} \left( \varepsilon + \sqrt{\frac{1 - t \varepsilon^2}{2}} \right) \]

\[ \leq 2\pi \left\{ e^2 + \frac{1}{2} \sqrt{1 - te^2} \right\} \leq 3\pi e^2, \]

\[ L^2(\partial D) \geq 4\pi^2 \{ t + (1 - t)e^2 \} \]

\[ \geq 4\pi^2 (1 - t)e^2 \geq 2\pi^2 e^2, \]

since \( 0 < t \leq 1/2. \) Thus, for \( 0 < t \leq e^2, \) \( L^2(\partial D)/A(D) \geq 2\pi/3, \) and so for rotationally symmetric domain \( D, \) we certainly have \( L^2(\partial D)/A(D) \geq 1/3. \)

**Step (ii).** Second, we consider the case where \( \gamma_1 \) and \( \gamma_2 \) may not be rotationally symmetric but neither of them intersects the pinching geodesic \( \gamma(t) = \{ (0, y) \in C_1 \} \) in \( C_1. \) Let \( \gamma' \) be the rotationally symmetric closed curve lying between \( \gamma_1, \gamma(t), \) and
touching $\gamma_1$. Let $D_1$ be the domain bounded by $\gamma_1$, $\gamma(t)$ and let $D'$ be the domain bounded by $\gamma'$, $\gamma(t)$. Then $D_1 \setminus D'$ is a union of several simply connected domains in $C_t$. By Theorem 2.7 and Lemma 2.8

$$(L(\gamma_1) + L'(\gamma'))^2 \geq 4\pi A(D_1 \setminus D').$$

On the other hand, since $D'$ is rotationally symmetric, by Step (i)

$$(L(\gamma') + L(\gamma(t)))^2 \geq \frac{2}{3} \pi A(D').$$

Since $L(\gamma_1) \geq L(\gamma')$, $L(\gamma_1) \geq L(\gamma(t))$,

$$8L^2(\gamma_1) \geq 4\pi A(D_1 \setminus D') + \frac{2}{3} \pi A(D') \geq \frac{2}{3} \pi A(D_1).$$

Similarly, let $D_2$ be the domain bounded by $\gamma_2$, $\gamma(t)$. Then $L^2(\gamma_2) \geq \pi A(D_2)/12$. Since $A(D) \leq A(D_1) + A(D_2)$ (if $\gamma_1$ and $\gamma_2$ lie on different sides of $\gamma(t)$, then the equality holds), and $L(\partial D) = L(\gamma_1) + L(\gamma_2)$,

$$L^2(\partial D) \geq L^2(\gamma_1) + L^2(\gamma_2) \geq \frac{\pi}{12} (A(D_1) + A(D_2)) \geq \frac{\pi}{12} A(D).$$

Step (iii). Third, we assume that only one of $\gamma_1$, $\gamma_2$ intersects $\gamma(t)$. Suppose $\gamma_1$ intersects $\gamma(t)$. Then the subdomain $D_1 = D \cap \{(x, y) \in C_t | x \leq 0\}$ of $D$ lying to the left of the pinching geodesic $\gamma(t)$ is a union of simply connected domains. Then by Theorem 2.7 and Lemma 2.8

$$(L(\gamma_1) + L(\gamma(t)))^2 \geq (L(\partial D_1))^2 \geq 4\pi A(D_1).$$

The right subdomain, $D_2 = D \cap \{(x, y) \in C_t | x \geq 0\}$, is contained in the domain $\tilde{D}_2$ bounded by $\gamma_2$, $\gamma(t)$. Since $\gamma_2$ does not intersect $\gamma(t)$, by Step (ii)

$$(L(\gamma_2) + L(\gamma(t)))^2 \geq \frac{\pi}{12} A(\tilde{D}_2) \geq \frac{\pi}{12} A(D_2).$$

Notice that for $i = 1, 2$, $L(\gamma_i) \geq L(\gamma(t))$; so

$$(L(\partial D))^2 \geq (L(\gamma_1))^2 + (L(\gamma_2))^2 \geq \frac{1}{4} \left\{(L(\gamma_1) + L(\gamma(t)))^2 + (L(\gamma_1) + L(\gamma(t)))^2\right\}.$$
Step (iv). Finally, we assume that both \( \gamma_1, \gamma_2 \) intersect \( \gamma(t) \). Then \( D \setminus \gamma(t) \) is a union of several simply connected domains. By Theorem 2.7 and Lemma 2.8
\[
(L(\gamma_1) + L(\gamma_2) + L(\gamma(t)))^2 \geq (L(\partial D \setminus \gamma(t)))^2 \geq 4\pi A(D \setminus \gamma(t)) = 4\pi A(D).
\]

Since \( L(\gamma_1) + L(\gamma_2) \geq 2L(\gamma(t)) \),
\[
(L(\partial D))^2 \geq \frac{1}{4} (L(\gamma_1) + L(\gamma_2) + L(\gamma(t)))^2 \geq \pi A(D).
\]

Combining cases (a) and (b), we get that, for \( 0 < t < \frac{1}{2} \), \( J(C) > \frac{\pi}{48} \). As stated at the beginning of the proof, for \( \frac{1}{2} < t < 1 \) the cone \( C_t \) forms a compact family of compact surfaces. Therefore, the proof of Proposition 2.6 is complete.

**Corollary 2.9.** For any conic degenerating family \( M_t \) of surfaces, if the pinching geodesic is nonseparating, then there exists a constant \( c > 0 \) depending only on the family such that, for \( 0 < t \leq 1 \),
\[
J(M_t) \geq c > 0.
\]

**Proof.** For the family \( M_t \), the complement of the pinching annulus \( C_t \) forms a compact family of compact surfaces. By assumption, the pinching geodesic \( \gamma(t) \) is nonseparating; thus, it suffices to consider the isoperimetric constants for the pinching annuli \( C_t \). Since the metrics on the pinching cones \( C_t \) are uniformly quasi-isometric to the standard metrics \( ds_t^2 \) on the cones \( C_t \) above and the isoperimetric constants are determined up to some multiple by the quasi-isometric class of the metrics, the corollary follows immediately from Proposition 2.6.

**Remark 2.10.** For a degenerating family \( M_t \) of surfaces with hyperbolic metrics, whether the pinching geodesics in \( M_t \) are separating or not, the isoperimetric constant of \( M_t \) (or of the pinching annulus) converges to zero as \( t \to 0 \). Because of this fact, the spectral degeneration for hyperbolic surfaces is more complicated. (See [Hj], [Ji], [W1], [W2].)

### 3. Spectral degeneration for cones

The proof of Theorem A is divided into two steps.

1. For all \( i \geq 1 \), \( \lim_{t \to 0} \lambda_t(i) \leq \lambda_i(0) \).
2. For all \( i \geq 1 \), \( \lim_{t \to 0} \lambda_t(i) \geq \lambda_i(0) \).

In this section we are going to prove step (1), that is, the following proposition.
**Proposition 3.1.** Given any conic degenerating family $M_t$ of surfaces,

$$\lim_{t \to 0} \lambda_i(t) \leq \lambda_i(0), \quad \text{for all } i \geq 1.$$ 

Let $M$ be any Riemannian surface with cone singularities. For simplicity we assume that $M$ has only one cone singular point, and unlike the degeneration model described in Section 2 we take it to be a “single” point, as opposed to a double point. Hence, we may write $M = K \cup C$, where $K$ is a compact, connected surface with boundary and $C = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 2\pi\}/\{(x, 0) \sim (x, 2\pi)\}$ endowed with a metric quasi-isometric to the standard one $ds^2 = dx^2 + x^2 dy^2$. For any $0 < \varepsilon < 1$ let $M_\varepsilon = K \cup \{(x, y) \in C|x \geq \varepsilon\}$ be the submanifold of $M$ obtained by cutting off a subcylinder.

Let $\{\lambda_i\}_i$ be all the eigenvalues of $M_t$ (counted with multiplicity) with respect to the Dirichlet boundary condition and let $\{\lambda_i\}_i$ be all the eigenvalues of $M$. Then we have the following proposition.

**Proposition 3.2.** For all $i \geq 1$, $\lim_{\varepsilon \to 0} \lambda_{i, \varepsilon} = \lambda_i$.

Before proving Proposition 3.2, we establish some lemmas whose statements and proofs are models for arguments later on. By a proof similar to that of Proposition 2.6, we immediately have the following lemma.

**Lemma 3.3.** For a surface $M$ with only cone singularities and subdomains $M_\varepsilon$ as above, there exists a constant $c > 0$ such that for all $0 < \varepsilon < 1$

$$\mathcal{F}(M_\varepsilon) \geq c > 0.$$ 

**Lemma 3.4.** For any sequence $\varepsilon_j \to 0$ let $\phi_{n_1, \varepsilon_j}, \ldots, \phi_{n_m, \varepsilon_j}$ be orthonormal eigenfunctions on $M_{\varepsilon_j}$ with eigenvalues $\lambda_{n_1, \varepsilon_j}, \ldots, \lambda_{n_m, \varepsilon_j}$. Assume that, for $1 \leq i \leq m$, $\lim_{\varepsilon_j \to 0} \lambda_{n_i, \varepsilon_j} = \lambda_{n_i}$ and $\phi_{n_i, \varepsilon_j}$ converges smoothly over compact subsets of $M$ to a function $\phi_{n_i}$ on $M$. Then the limit functions $\phi_{n_1}, \ldots, \phi_{n_m}$ are orthonormal eigenfunctions of $M$ with eigenvalues $\lambda_{n_1}, \ldots, \lambda_{n_m}$.

**Proof.** First of all, it is clear that for $1 \leq i \leq m$

$$(\Delta - \lambda_n)\phi_{n_i} = 0, \quad \|\phi_{n_i}\|_{L^2} \leq 1$$

and

$$\|\nabla \phi_{n_i}\|_{L^2} \leq \lambda_{n_i} < +\infty$$

since $\|\phi_{n_i, \varepsilon_j}\|_{L^2} = 1$ and $\|\nabla \phi_{n_i, \varepsilon_j}\|_{L^2} = \lambda_{n_i, \varepsilon_j}$. Therefore, it suffices to prove that for $1 \leq i, k \leq m$

$$\langle \phi_{n_i}, \phi_{n_k}\rangle = \delta_{ik}.$$ 

For any $0 < \varepsilon < \delta < 1$ define a subdomain $C_{\varepsilon, \delta} = \{(x, y) \in C|x \leq \varepsilon \leq \delta\} \subset C \subset M$ and $C_\delta = C_{0, \delta}$ a subcylinder of $C$. It is clear that $C_{\varepsilon, \delta} \subset C_\delta$. Then for any $0 < \varepsilon_j <
\(\delta < 1\) and any \(1 \leq i, k \leq m,\)

\[
\langle \phi_{n_i}, \phi_{n_k} \rangle_M = \int_M \phi_{n_i} \phi_{n_k} = \int_{M \setminus C_3} \phi_{n_i} \phi_{n_k} + \int_{C_3} \phi_{n_i} \phi_{n_k},
\]

\(\delta_{ik} = \langle \phi_{n_i, \varepsilon_j}, \phi_{n_k, \varepsilon_j} \rangle_{M_{\varepsilon_j}} = \int_{M \setminus C_3} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j} + \int_{C_3} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j}.
\]

Therefore,

\[
|\langle \phi_{n_i}, \phi_{n_k} \rangle_M - \delta_{ik}| \leq \left| \int_{M \setminus C_3} \phi_{n_i} \phi_{n_k} - \int_{M \setminus C_3} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j} \right| + \left| \int_{C_3} \phi_{n_i} \phi_{n_k} + \int_{C_3} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j} \right|.
\]

By Theorems 2.4, 2.5, and Lemma 3.3, \(\phi_{n_i, \varepsilon_j}, \ldots, \phi_{n_m, \varepsilon_j}\) are bounded from above independent of \(\{\varepsilon_j\}; \phi_{n_1, \ldots, \phi_{n_n}}\) are therefore bounded as well by the assumption of convergence of \(\{\phi_{n_i, \varepsilon_j}\}\) and the uniform bound on the latter. Furthermore, \(\lim_{k \to 0} A(C_\delta) = \lim_{\delta \to 0} \lim_{i \to 0} A(C_{\varepsilon_j, \delta}) = 0\). Then for any \(\delta' > 0\) there exists \(\delta_0 > 0\) such that for \(0 < \delta \leq \delta_0\)

\[
\left| \int_{C_3} \phi_{n_i} \phi_{n_k} + \lim_{\varepsilon_j \to 0} \int_{C_{\varepsilon_j, \delta}} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j} \right| \leq \delta'.
\]

Notice the uniform convergence of \(\phi_{n_i, \varepsilon_j}, \phi_{n_k, \varepsilon_j}\) to \(\phi_{n_i}, \phi_{n_k}\), respectively, over compact subsets of \(M\). It follows that

\[
|\langle \phi_{n_i}, \phi_{n_k} \rangle_M - \delta_{ik}| \leq \lim_{\varepsilon_j \to 0} \int_{M \setminus C_3} \phi_{n_i} \phi_{n_k} - \int_{M \setminus C_3} \phi_{n_i, \varepsilon_j} \phi_{n_k, \varepsilon_j} + \delta' = \delta'.
\]

Since \(\delta' > 0\) is arbitrary, for \(1 \leq i, k \leq n,\) we have

\[
\langle \phi_{n_i}, \phi_{n_k} \rangle_M - \delta_{ik} = 0.
\]
Lemma 3.5. Given any sequence \( \varepsilon_j \to 0 \) and a normalized eigenfunction \( \phi_{\varepsilon_j} \) on \( M_{\varepsilon_j} \) with eigenvalue \( \lambda_{\varepsilon_j} \), assume that \( \lim_{\varepsilon_j \to 0} \lambda_{\varepsilon_j} < +\infty \). Then there exists a subsequence \( \varepsilon_j' \to 0 \) such that \( \lim_{\varepsilon_j' \to 0} \lambda_{\varepsilon_j'} \) exists and \( \phi_{\varepsilon_j} \) converges smoothly over compact subsets of \( M \) to a function \( \varphi \), which is a normalized eigenfunction on \( M \) with eigenvalue \( \lim_{\varepsilon_j' \to 0} \lambda_{\varepsilon_j'} \).

Proof. We have

\[
(\Delta - \lambda_{\varepsilon_j})\phi_{\varepsilon_j} = 0 \quad \text{and} \quad \int_{M_{\varepsilon_j}} |\nabla \phi_{\varepsilon_j}|^2 = \lambda_{\varepsilon_j} \int_{M_{\varepsilon_j}} |\phi_{\varepsilon_j}|^2 = \lambda_{\varepsilon_j}.
\]

By regularity theory (see Theorems 8.8, 8.10 in [GT]), for any compact subset \( \bar{D} \subset M \) and any \( k \in \mathbb{N} \), there exists a constant \( c = c(\bar{D}, \lim_{\varepsilon_j \to 0} \lambda_{\varepsilon_j}) \) such that

\[
\|\phi_{\varepsilon_j}\|_{W^{k,2}(\bar{D})} \leq c.
\]

Then by the Sobolev embedding theorem (see Theorem 5.4 in [Ad]) and a diagonal argument, there exists a subsequence \( \varepsilon_j' \to 0 \) such that \( \lim_{\varepsilon_j' \to 0} \lambda_{\varepsilon_j} \) exists and \( \phi_{\varepsilon_j} \) converges over compact subsets of \( M \) to a function \( \varphi \). By Lemma 3.4 the limit function \( \varphi \) is a normalized eigenfunction on \( M \) with eigenvalue \( \lim_{\varepsilon_j' \to 0} \lambda_{\varepsilon_j'} \).

Proof of Proposition 3.2. By domain monotonicity for Dirichlet eigenvalues, \( \lambda_{i,\varepsilon} \leq \lambda_{i,1/2} \) for \( 0 < \varepsilon < 1/2 \) and \( i \geq 1 \). For any sequence \( \varepsilon_j \to 0 \) let \( \{\psi_{i,\varepsilon_j}\}_{1}^{\infty} \) be a complete system of orthonormal eigenfunctions with eigenvalues \( \{\lambda_{i,\varepsilon_j}\}_{1}^{\infty} \). By Lemmas 3.4, 3.5, and a diagonal argument, there exists a subsequence \( \varepsilon_j' \to 0 \) such that, for all \( i \geq 1 \), \( \lambda_i^{*} = \lim_{\varepsilon_j' \to 0} \lambda_{i,\varepsilon_j} \) exists, \( \psi_{i,\varepsilon_j} \) converges smoothly over compact subsets of \( M \) to an eigenfunction \( \phi_i^{*} \) on \( M \) with eigenvalues \( \lambda_i^{*} \), and the limit functions \( \{\phi_i^{*}\}_{1}^{\infty} \) are orthonormal.

Claim. The limit functions \( \{\phi_i^{*}\}_{1}^{\infty} \) form a complete system of orthonormal eigenfunctions with eigenvalues \( \{\lambda_i^{*}\}_{1}^{\infty} \).

Assuming the claim, it is clear that, for all \( i \geq 1 \), \( \lambda_i^{*} = \lambda_i \). By the arbitrary choice of \( \varepsilon_j \to 0 \), for all \( i \geq 1 \)

\[
\lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} = \lambda_i.
\]

This completes the proof of Proposition 3.2.

Proof of claim. Assume the contrary. Then there exists a normalized eigenfunction \( \varphi \) on \( M \) with eigenvalue \( \lambda \) such that, for all \( i \geq 1 \), \( \langle \varphi, \phi_i^{*} \rangle = 0 \). Let \( \eta_{\varepsilon} = \eta_{\varepsilon}(x) \) be a cutoff function on \( M \) such that \( \eta_{\varepsilon} = 1 \) on \( M_{3\varepsilon} = M \setminus C_{3\varepsilon} \), \( \eta_{\varepsilon} = 0 \) on \( C_{2\varepsilon} \), and
$|\nabla \eta| \leq 2/\varepsilon$. Then

$$\lim_{\varepsilon \to 0} \int_M |\varphi \eta| \leq \int_M |\varphi|^2 = 1$$

and

$$\int_M |\nabla (\varphi \eta)|^2 \leq \int_M |\nabla \varphi|^2 + \int_M |\varphi|^2 ||\nabla \eta||^2$$

$$\leq \lambda \int_M |\varphi|^2 + \frac{4}{\varepsilon^2} \int_{C_{i,2r}} |\varphi|^2.$$

By Theorems 2.4, 2.5, and Lemma 3.3, there exists some constant $e_0 > 0$ such that $\|\varphi\|_{L^\infty} \leq e_0 < +\infty$. Then

$$\int_{C_{i,2r}} |\varphi|^2 \leq e_0 \int_{C_{i,2r}} 1$$

$$= e_0 A(C_{i,2r}) = \frac{3}{2} e_0 \varepsilon^2.$$

Therefore,

$$\int_M |\nabla (\varphi \eta)|^2 \leq \lambda + 6e_0. \quad (3.6)$$

Expanding the function $\varphi \eta_{i,j}$ in terms of the complete system of orthonormal eigenfunctions $\{\varphi_{i,j}\}_{i,j}$ on $M_{i,j}$,

$$\varphi \eta_{i,j} = \sum_{i=1}^{\infty} a_i(e_j) \varphi_{i,j} \quad (3.7)$$

where for $i \geq 1$

$$a_i(e_j) = \langle \varphi \eta_{i,j}, \varphi_{i,j} \rangle_{M_{i,j}}$$

and

$$\sum_{i=1}^{\infty} a_i^2(e_j) = \|\varphi \eta_{i,j}\|^2.$$
where the inequality $\lambda_{i,e_j} \geq \lambda_i$ follows from the domain monotonicity for Dirichlet eigenvalues. For any $N \in \mathbb{N}$

$$\|\nabla (\varphi \eta_j)\|^2 \geq \lambda_N \sum_{i \geq N} a_i^2(e_j).$$

Since $\lambda_N \to +\infty$ as $N \to +\infty$, by equation 3.6, for any $0 < \delta < 1$ there exists $N_0$ independent of $e_j$ such that

$$\sum_{i > N_0} a_i^2(e_j) \leq \delta \quad \text{and} \quad \sum_{i=1}^{N_0} a_i^2(e_j) \geq \|\varphi \eta_j\|^2 - \delta. \tag{3.8}$$

On the other hand, for $1 \leq i \leq N_0$

$$\lim_{e_j \to 0} a_i(e_j) = \lim_{e_j \to 0} \langle \varphi \eta_j, \varphi_{i,e_j} \rangle_{M_j} = \langle \varphi, \varphi_i^* \rangle = 0$$

where in the second equality, we use the fact that $\varphi_{i,e_j}$ is bounded independent of $e_j$; this follows from Theorems 2.4, 2.5, Lemma 3.3, and was used in the proof of Lemma 3.4. Then letting $e_j \to 0$ in equation 3.8, we get

$$0 = \lim_{e_j \to 0} \sum_{i=1}^{N_0} a_i^2(e_j) \geq \lim_{e_j \to 0} \|\varphi \eta_j\|^2 - \delta = \|\varphi\|^2 - \delta = 1 - \delta.$$

Since $\delta < 1$, this is a contradiction! Thus, we have proven the claim and thence Proposition 3.2.

Remark 3.9. The basic philosophy here is that, since in the limiting process of $M_\epsilon \to M$ as $\epsilon \to 0$ no mass of the eigenfunctions of $M_\epsilon$ is lost (see Lemma 3.4), it is reasonable that all eigenfunctions on $M$ should come from eigenfunctions on $M_\epsilon$.

Remark 3.10. A special case of Proposition 3.2 and its proof is the following fact. (See [CF1].) Let $M^n$ be a compact Riemannian manifold of dimension $n \geq 2$, $p$ be a distinguished point in $M$, and $M_\epsilon$ (for $\epsilon > 0$ small) be the complement of the geodesic ball around $p$ with radius $\epsilon$. Then the Dirichlet eigenvalues of $M_\epsilon$ converge to eigenvalues of $M$ as $\epsilon \to 0$.

Proof of Proposition 3.1. For $1 > \epsilon > 0$ let $M_{t,\epsilon} = M \setminus \{(x, y) \in C_t \mid |x| < \epsilon\}$ and let $\{\lambda_i(t, \epsilon)\}_{i=1}^\infty$ be all the eigenvalues of $M_{t,\epsilon}$ with respect to the Dirichlet boundary
condition. Then by the domain monotonicity for eigenvalues, for all \( i \geq 1 \)

\[
\lambda_i(t) \leq \lambda_i(t, \varepsilon).
\]

For any fixed \( \varepsilon > 0 \), \( M_{t, \varepsilon} (0 \leq t \leq 1) \) forms a compact family of compact surfaces. Thus, for all \( i \geq 1 \)

\[
\lim_{t \to 0} \lambda_i(t, \varepsilon) = \lambda_i(0, \varepsilon).
\]

By Proposition 3.2, for all \( i \geq 1 \),

\[
\lim_{t \to 0} \lambda_i(t, \varepsilon) = \lambda_i(0, \varepsilon).
\]

Therefore, for all \( i \geq 1 \)

\[
\lim_{t \to 0} \lambda_i(t, \varepsilon) \leq \lim_{t \to 0} \lambda_i(0, \varepsilon) = \lambda_i(0).
\]

This completes the proof.

To prepare for the proof of \( \lim_{t \to 0} \lambda_i(t) \geq \lambda_i(0) \) \( (i \geq 1) \) in the next section, we study the spectral degeneration for the cone family \( C_t \) first. With respect to the Dirichlet boundary condition on \( \partial C_t \), let \( \{\mu_i(t)\}_{i=1}^{\infty} \) be all the eigenvalues of \( C_t \) counted with multiplicity.

**Proposition 3.11**

(i) For all \( i \geq 1 \), \( \lim_{t \to 0} \mu_i(t) = \mu_i(0) \).

(ii) For any sequence \( t_j \to 0 \) let \( \{\psi_i(t_j)\}_{i=1}^{\infty} \) be a complete system of orthonormal (Dirichlet) eigenfunctions on \( C_{t_j} \) with eigenvalues \( \{\mu_i(t_j)\}_{i=1}^{\infty} \). Then there exists a subsequence \( t_j' \to 0 \) such that, for all \( i \geq 1 \), \( \psi_i(t_j') \) converges smoothly over compact subsets of \( C_0 \) to an eigenfunction \( \psi_i(0) \) with eigenvalue \( \mu_i(0) \), and \( \{\psi_i(0)\}_{i=1}^{\infty} \) is a complete system of orthonormal Dirichlet eigenfunctions on \( C_0 \).

**Proof:** By the same proof as that of Proposition 3.1, for all \( i \geq 1 \)

\[
\lim_{t \to 0} \mu_i(t) \leq \mu_i(0).
\]

On the other hand, from Proposition 2.6 there exists a constant \( c > 0 \) such that, for \( 0 \leq t \leq 1 \), \( \mathcal{F}(C_t) \geq c > 0 \). Then by arguments similar to those in the proofs of Lemmas 3.3 and 3.4, there exists a subsequence \( t_j' \to 0 \) such that, for all \( i \geq 1 \), \( \psi_i(t_j') \) converges smoothly over compact subsets of \( C_0 \) to a Dirichlet eigenfunction \( \psi_i(0) \) with eigenvalue \( \mu_i(0) \), and \( \{\psi_i(0)\}_{i=1}^{\infty} \) are orthonormal Dirichlet eigenfunctions on \( C_0 \). It is clear then that for all \( i \geq 1 \)

\[
\lim_{t_j' \to 0} \mu_i(t_j') \geq \mu_i(0).
\]
By the arbitrary choice of $t_j \to 0$

$$\lim_{t \to 0} \mu_i(t) \geq \mu_i(0).$$

Therefore, for all $i \geq 1$

$$\lim_{t \to 0} \mu_i(t) = \mu_i(0),$$

and \{$\psi_i(0)$\}$_{i=1}^\infty$ is a complete system of orthonormal Dirichlet eigenfunctions on $C_0$ with eigenvalues \{$\mu_i(0)$\}$_{i=1}^\infty$.

**Remark 3.12.** By Corollary 2.9, if the pinching geodesic $\gamma(t)$ in the conic family $M_t$ is nonseparating, then the isoperimetric constant $\mathcal{I}(M_t) \geq c > 0$ for some constant $c$ independent of $t$, and the proof above works for Theorem A in this case also. But for the case of the separating pinching geodesic, we need another argument. Instead, in Section 4 we prove Theorem A simultaneously for the pinching geodesic separating or not, thus justifying the philosophy that, to understand general degenerating families, it suffices to understand the degeneration of the pinched part. (see [Ji].)

**Remark 3.13.** The above proof for Proposition 3.11 gives a new, elementary proof of Theorem B in [CF2].

### 4. Proof of Theorem A.

In this section we prove that for all $i \geq 1$

$$\lim_{t \to 0} \lambda_i(t) \geq \lambda_i(0),$$

and we finish the proof of Theorem A.

By Proposition 3.1, for all $i \geq 1$

$$\lim_{t \to 0} \lambda_i(t) \leq \lambda_i(0).$$

Then by arguments similar to those in the proofs of Lemmas 3.3 and 3.4, for any sequence $t_j \to 0$ there exists a subsequence $t_j' \to 0$ such that, for all $i \geq 1$, $\varphi_i(t_j')$ converges smoothly over compact subsets of $M_0$ to a function $\varphi_i^*$ on $M_0$, and $\lambda_i^* = \lim_{t \to 0} \lambda_i(t_j')$ exists. The limit function $\varphi_i^*$ satisfies

$$(\Delta - \lambda_i^*) \varphi_i^* = 0, \quad \|\varphi_i^*\|_{L^2} \leq 1 \quad \text{and} \quad \|\nabla \varphi_i^*\|_{L^2} \leq \lambda_i^* < +\infty.$$  

**Lemma 4.1.** The limit functions \{$\varphi_i^*$\}$_{i=1}^\infty$ are orthonormal eigenfunctions on $M_0$ with eigenvalues \{$\lambda_i^*$\}$_{i=1}^\infty$, that is, for all $i, k \geq 1$

$$\langle \varphi_i^*, \varphi_k^* \rangle = \delta_{ik}.$$
Assume Lemma 4.1 first. Then the limit functions \( \{\varphi_i^*\}_{i=1}^\infty \) are, in particular, linearly independent, and thus by min-max we have for all \( i \geq 1 \)
\[
\lim_{t_j \to 0} \lambda_i(t_j) = \lambda_i^* \geq \lambda_i(0).
\]
By the arbitrary choice of the sequence \( t_j \to 0 \), for all \( i \geq 1 \)
\[
\lim_{t \to 0} \lambda_i(t) \geq \lambda_i(0).
\]
Therefore, combined with Proposition 3.1,
\[
\lim_{t \to 0} \lambda_i(t) = \lambda_i(0), \quad \text{for all } i \geq 1.
\]
The limit functions \( \{\varphi_i^*\}_{i=1}^\infty \) are a complete system of orthonormal eigenfunctions on \( M_0 \) with eigenvalues \( \{\lambda_i(0)\}_{i=1}^\infty \) and thus may be denoted by \( \{\varphi_i(0)\}_{i=1}^\infty \). This proves parts (i) and (ii) of Theorem A. For part (iii), \( \lambda_i(0) < \lambda \notin \text{Spec}(A_0) \) if and only if \( \lambda_i(t) < \lambda \) for small \( t \); so by parts (i) and (ii)
\[
\lim_{t_j \to 0} K_{t_j}(x, y; \lambda) = K_0(x, y; \lambda).
\]
By the arbitrary choice of the sequence \( t_j \to 0 \),
\[
\lim_{t \to 0} K_t(x, y; \lambda) = K_0(x, y; \lambda).
\]
This completes the proof of Theorem A under the assumption of Lemma 4.1.

**Proof of Lemma 4.1.** Let \( \eta = \eta(x) \) be a cutoff function on \( M_0 \) such that \( \eta = 0 \) on \( M_{0,1/2} = M_0 \setminus \{(x, y) \in C_0 ||x| \leq 1/2\} \), \( \eta = 1 \) on \( M_0 \setminus M_{0,1/4} = \{(x, y) \in C_0 ||x| \geq 1/4\} \), and \( |\nabla \eta| \leq 8 \) on \( M_0 \). For any fixed \( i_0 \geq 1 \) consider the function \( \varphi_{i_0}(t_j)\eta \) on \( C_{t_j} \).

We want to show that \( \varphi_{i_0}(t_j)\eta \) does not lose any mass inside the pinching annulus during degeneration. More precisely, define
\[
m_0 = \lim_{t_j \to 0} \int_{M_{t_j}} |\varphi_{i_0}(t_j)\eta|^2;
\]
then we have the following claim.

**Claim.** (i) The mass of the limit function \( \varphi_{i_0}^*\eta \) is
\[
\int_{M_0} |\varphi_{i_0}^*\eta|^2 = m_0;
\]
(ii) The mass lost during degeneration is

$$\lim_{\varepsilon \to 0} \lim_{t_j \to 0} \int_{M_{t_j} \setminus M_{t_j - \varepsilon}} |\varphi_{t_j}(t_j)|^2 = 0.$$ 

**Proof of claim.** Actually, by taking a subsequence, if necessary, we may assume that $\lim_{t_j \to 0} \int_{M_{t_j}} |\varphi_{t_j}(t_j)\eta|^2 = m_0$. Expanding the function $\varphi_{t_j}(t_j)\eta$ on $C_{t_j}$ in terms of a complete system of orthonormal Dirichlet eigenfunctions $\{\psi_i(t_j)\}_{i=1}^\infty$ on $C_{t_j}$ with eigenvalues $\{\mu_i(t_j)\}_{i=1}^\infty$,

$$\varphi_{t_j}(t_j)\eta = \sum_{i=1}^\infty b_i(t_j)\psi_i(t_j)$$

where

$$\sum_{i=1}^\infty b_i^2(t_j) = \int_{M_{t_j}} |\varphi_{t_j}(t_j)\eta|^2,$$

$$\sum_{i=1}^\infty b_i^2(t_j)\mu_i(t_j) = \int_{M_{t_j}} |\nabla \varphi_{t_j}(t_j)\eta|^2.$$ 

Now

$$\int_{M_{t_j}} |\nabla \varphi_{t_j}(t_j)\eta|^2 \leq \int_{M_{t_j}} |\nabla \varphi_{t_j}(t_j)|^2 + \max |\nabla \eta| \int_{M_{t_j}} |\varphi_{t_j}(t_j)|^2$$

$$\leq \lambda_{t_j}(t_j) + 8 \leq c_0$$

for some constant $c_0 < +\infty$ independent of $t_j$, using $\lim_{t_j \to 0} \lambda_{t_j}(t) \leq \lambda_{t_j}(0)$. Thus, for any $N \in \mathbb{N}$

$$\mu_{N}(t_j) \sum_{i \geq N} b_i^2(t_j) \leq \sum_{i \geq N} b_i^2(t_j) \mu_i(t_j) \leq \sum_{i=1}^\infty b_i^2(t_j) \mu_i(t_j)$$

$$= \int_{M_{t_j}} |\nabla \varphi_{t_j}(t_j)\eta|^2 \leq c_0.$$ 

Since $\lim_{t \to 0} \mu_{N}(t) = \mu_N(0)$ and $\lim_{N \to +\infty} \mu_N(0) = +\infty$, for any $\delta > 0$ there exists an
$N_0 \in \mathbb{N}$ such that

$$\sum_{i > N_0}^\infty b_i^2(t_j) \leq \delta,$$

$$\sum_{i=1}^{N_0} b_i^2(t_j) \geq \int_{M_{t_j}} |\varphi_{t_0}(t_j)\eta|^2 - \delta,$$

$$\int_{C_{t_j}} \left| \varphi_{t_0}(t_j)\eta - \sum_{i=1}^{N_0} b_i^2(t_j)\psi_i(t_j) \right|^2 \leq \sum_{i > N_0}^\infty b_i^2(t_j) \leq \delta.$$ 

By further taking a subsequence we assume that, for $1 \leq i \leq N_0$, $\lim_{t_j \to 0} b_i(t_j) = b_i(0)$. Then

$$\sum_{i=1}^{N_0} b_i^2(0) \geq \lim_{t_j \to 0} \sum_{i=1}^{N_0} b_i^2(t_j)$$

$$\geq \lim_{t_j \to 0} \int_{M_{t_j}} |\varphi_{t_0}(t_j)\eta|^2 - \delta = m_0 - \delta.$$ 

For any $1 > \varepsilon > 0$ and $0 \leq t \leq 1$, let $C_{\varepsilon} = \{(x, y) \in C_{t_j} \mid |x| \geq \varepsilon \}$. We then have

$$\int_{C_{\varepsilon}} |\varphi_{t_0}\eta|^2 = \lim_{t_j \to 0} \int_{C_{t_j}} |\varphi_{t_0}(t_j)\eta|^2$$

$$\geq \lim_{t_j \to 0} \int_{C_{t_j}} \left| \sum_{i=1}^{N_0} b_i(t_j)\psi_i(t_j) \right|^2$$

$$- \lim_{t_j \to 0} \int_{C_{t_j}} \left| \varphi_{t_0}(t_j)\eta - \sum_{i=1}^{N_0} b_i(t_j)\psi_i(t_j) \right|^2$$

$$\geq \lim_{t_j \to 0} \int_{C_{t_j}} \left| \sum_{i=1}^{N_0} b_i(t_j)\psi_i(t_j) \right|^2 - \delta$$

$$- \lim_{t_j \to 0} \int_{C_{t_j} \setminus C_{\varepsilon}} \left| \sum_{i=1}^{N_0} b_i(t_j)\psi_i(t_j) \right|^2 - \delta$$

$$\geq \lim_{t_j \to 0} \sum_{i=1}^{N_0} b_i^2(t_j) - N_0 \sum_{i=1}^{N_0} b_i^2(0) \lim_{t_j \to 0} \int_{C_{t_j} \setminus C_{\varepsilon}} |\psi_i(t_j)|^2 - \delta.$$
By Proposition 3.11, \( \|\psi_i(t')\|_{L^2} = \|\psi_i(0)\|_{L^2} = 1 \), and \( \psi_i(t') \) converges to \( \psi_i(0) \) uniformly over compact subsets of \( C_0 \); so

\[
\lim_{\epsilon \to 0} \lim_{t' \to 0} \int_{C_{y'} \setminus C_{y}(\epsilon)} |\psi_i(t')|^2 = \lim_{\epsilon \to 0} \int_{C_0 \setminus C_{y}(\epsilon)} |\psi_i(0)|^2 = 0.
\]

Thus, for any \( \delta > 0 \) given above, there exists \( \epsilon_0 > 0 \) such that for \( \epsilon < \epsilon_0 \)

\[
N_0 \sum_{i=1}^{N_0} b_i^2(0) \lim_{t' \to 0} \int_{C_{y'} \setminus C_{y}(\epsilon)} |\psi_i(t')|^2 \leq \delta.
\]

And for any \( \delta > 0 \) and \( \epsilon < \epsilon_0 < 1/4 \)

\[
\int_{C_0} |\varphi_i^* \eta|^2 \geq \int_{C_{y}(\epsilon)} |\varphi_i^* \eta|^2 \geq m_0 - 2\delta.
\]

Since \( \delta > 0 \) is arbitrary,

\[
\int_{C_0} |\varphi_i^* \eta|^2 = m_0.
\]

This completes part (i) of the claim. For part (ii) we have for \( \epsilon < \epsilon_0 < 1/4 \)

\[
\lim_{t' \to 0} \int_{M_{y'} \setminus M_{y'}} |\varphi_{i_0}(t')|^2 = \lim_{t' \to 0} \int_{C_{y'} \setminus C_{y}(\epsilon)} |\varphi_{i_0}(t')|^2 \\
\leq \lim_{t' \to 0} \int_{C_{y}} |\varphi_{i_0}(t')\eta|^2 - \lim_{t' \to 0} \int_{C_{y}(\epsilon)} |\varphi_{i_0}(t')\eta|^2 \\
= \lim_{t' \to 0} \int_{C_{y}} |\varphi_{i_0}(t')\eta|^2 - \int_{C_{y}(\epsilon)} |\varphi_i^* \eta|^2 \\
\leq m_0 - (m_0 - 2\delta) = 2\delta.
\]

Since \( \delta > 0 \) is arbitrary, part (ii) of the claim follows immediately, and the proof of the claim is complete.

We now use the claim to prove the orthonormality of the limit functions \( \{\varphi_{i_0}^*\}_{i=1}^{\infty} \). Combining this with the equations preceding Lemma 4.1, we will have shown that the limit functions \( \{\varphi_{i_0}^*\}_{i=1}^{\infty} \) are orthonormal eigenfunctions on \( M_0 \) with eigenvalues \( \{\lambda_i^*\}_{i=1}^{\infty} \).
For any $i, k \geq 1$,

$$|\langle \phi_i^*, \phi_k^* \rangle - \delta_{ik}| = \lim_{\epsilon \to 0} \left| \int_{M_{0,\epsilon}} \phi_i^* \phi_k^* - \delta_{ik} \right|$$

$$= \lim_{\epsilon \to 0} \lim_{t_j \to 0} \left| \int_{M_{t_j,\epsilon}} \phi_i(t_j) \phi_k(t_j) - \delta_{ik} \right|$$

$$\leq \lim_{\epsilon \to 0} \lim_{t_j \to 0} \left( \int_{M_{t_j,\epsilon}} |\phi_i(t_j)|^2 \right)^{1/2} \left( \int_{M_{t_j,\epsilon}} |\phi_k(t_j)|^2 \right)^{1/2}$$

$$= 0$$

where in the last equality we use part (ii) of the claim. This completes the proof of Lemma 4.1.

5. Proof of Theorem B. In this section and the next, we present examples of metrics defined on Riemann surfaces whose spectra converge by the results of the previous sections. As we have seen, the key estimate needed is a lower bound on the isoperimetric constant.

We shall consider a specific construction of an analytic family of Riemann surfaces $\pi: \mathcal{M} \to D$, where $D$ is the unit disk in $\mathbb{C}$. This construction is standard, and we refer to [F1] for more details. Briefly, there are two cases to consider: (i) we start with two compact Riemann surfaces $M_1, M_2$ of genus $g_1 = g - j, g_2 = j$ respectively (we always take $j \leq g/2, g \geq 2$), and local coordinates $z_1, z_2$ centered at points $p_1 \in M_1, p_2 \in M_2$. For $t \in D \setminus \{0\}$ remove the disks $|z_1| < |t|$ and glue together the remaining surfaces by means of the identification $z_1 = z_2 = t$. The resulting surfaces may be completed to form an analytic family $\pi: \mathcal{M} \to D$, where $M_t = \pi^{-1}(t)$ has genus $g$ for $t \neq 0$ and $\pi^{-1}(0)$ is stable in the sense of Deligne-Mumford. Alternately, (ii) we could start with a single surface $M$ of genus $g = 1 > 0$ and coordinates about two points $a, b \in M$ and a similar construction adds a handle to $M$. Then the fiber $\pi^{-1}(t)$ would have genus $g = 0$. In both cases (i) and (ii), we shall use the notation $M_0 = \pi^{-1}(0)$ and denote the identified double point (or "node") by $p$. The two types of degeneration are distinguished, as discussed in Section 2, by whether $p$ separates the degenerate surface $M_0$.

We also introduce some notation: let $U_t = \{q \in M_1 ||z_1(q)|| < |t|^{1/2}\}$ and suppose $R$ is any region in $M_1$. Then there is a natural embedding $R \setminus U_t \to M_t$ under the identification described above. We shall denote the image $R \cap M_t$. This works as well for $R \subset M_2$ or $R \subset M$ in the nonseparating case. If $R$ is, for example, an open submanifold of $M_1$ and $\bar{R} \subset M_1 \setminus \{p\}$, then for small $|t|$, $R$ is embedded in $M_t$, and
a metric $ds^2$ on $M$, may be pulled back via this embedding and compared to a fixed
metric on $R \subset M_1$. The estimates in the following two sections should be taken in
this sense.

We now proceed to define the Bergman metric: let $M$ be a compact Riemann
surface of genus $g > 0$. Let $\omega_1, \ldots, \omega_g$ be a basis of abelian differentials, normalized
with respect to the $A$-cycles of some symplectic homology basis for $M$, and denote
by $\Omega_{ij}$ the associated period matrix.

**Definition 5.1.** The Bergman metric for $M$ is defined by $ds^2 = \mu(z)|dz|^2$, where

$$
\mu(z) = \frac{1}{g} \sum_{i,j=1}^{g} (\text{Im} \Omega_{ij}^{-1}) \omega_i(z) \overline{\omega_j(z)}.
$$

**Remark 5.2.** The Riemann surface $M$ may be embedded into a $g$-dimensional
complex torus $J(M)$, called the Jacobian variety on $M$. The metric $\mu$ is the one
induced by this embedding from the natural Euclidean metric on $J(M)$. Since the
scalar curvature of subvarieties decreases, we know that the scalar curvature of $\mu$
is nonpositive. (See [GH], p. 79.)

Now suppose we consider the Bergman metrics $\mu_i$ on the degenerating family $\mathcal{M}$
described above.

**Proposition 5.3** ([W], Lemmas 6.9 and 7.4).

(i) For the degeneration (i) described above

$$
\mu_i \to \frac{g_i}{g} \mu_i
$$

uniformly on compact subsets of $M_i \setminus \{p_i\}$, $i = 1, 2$, where $\mu_i$ is the Bergman metric of
$M_i$. Moreover, there is a constant $C$ depending only on the family such that, in local
coordinates about the node,

$$
\left| \frac{\mu_i(z) - g_i}{g} \mu_i(z) \right| \leq C |t|/|z|^2.
$$

(ii) For the degeneration (ii) described above

$$
\mu_i \to \frac{g - 1}{g} \mu
$$

uniformly on compact subsets of $M_0 \setminus \{p\}$. Moreover, in local coordinates about the
node,

$$
\left| \frac{\mu_i(z) - g - 1}{g} \mu(z) - \frac{1}{-\log|t|} \frac{1}{2\pi g |z|^2} \right| \leq O \left( \frac{1}{-\log|t|} \right)
$$
where the estimate is

$$
\lim_{t \to 0} \sup_{|t|^{1/2} < |z|} \left( -|z|^2 \log|t| \right) O\left( \frac{1}{-\log|t|} \right) = 0.
$$

Fix a geodesic disk \( C \) about the node \( p \) in \( M_0 \). Then for \( t \neq 0 \), \( C_t = C \cap M_t \) is topologically a cylinder which contains the pinching region.

**Corollary 5.4.** Let \( M_t \) be degenerating to a separating node, where \( M_t \) is equipped with the Bergman metric and \( C_t \) is as above. Then there exists a constant \( c \) depending only on \( M \) such that for all \( t \in D \setminus \{0\} \)

$$
\mathcal{F}(C_t) \geq c > 0.
$$

**Proof.** By Remark 5.2 and Theorem 2.7, we may restrict our attention to homotopically nontrivial curves, rotationally symmetric as in Section 2. By Proposition 5.3, part (i), the error in estimating the lengths of such curves by the limiting metrics vanishes as \( t \to 0 \), and the corresponding subdomains clearly have finite area. Thus, \( \mathcal{F}(C_t) \) may be bounded below by \( \mathcal{F}(C_0) \) for the limiting metrics, which is clearly bounded away from zero.

**Proof of Theorem B, part (i).** The proof proceeds exactly as in Section 4, the crucial point being the bound of Corollary 5.4 and the discreteness of the spectrum for the limiting metric, which in this case is obvious. Note that by Remark 3.10 the limiting spectrum is indeed the spectrum for the closed problem on the disjoint union of \( M_1 \) and \( M_2 \) with a multiple of the Bergman metric.

As noted in the introduction, the pinching region for the nonseparating case becomes long and thin. This is easily seen from the result in part (ii) of Proposition 5.3. In order to prove part (ii) of Theorem B, we wish to compare the Bergman metric to one where the long, thin cylinder is actually flat. Set \( l = \sqrt{-\log|t|} \) and construct a family of interpolating metrics \( \tilde{\mu}_t \) satisfying

1. \( \tilde{\mu}_t = \mu_t \) on the complement of the pinching annulus \( \{z||z| < l^{-1}\} \);
2. \( \tilde{\mu}_t (z) = (-\log|t|)^{1/2} |z|^{-1} \), for \( |z| < \frac{1}{2} l^{-1} \);
3. \( \sup_{1/2 < |z| < l^{-1}} (\tilde{\mu}_t(z)) \) is bounded independently of \( t \).

Now choose \( L > 0 \), also independent of \( t \), such that for \( t \neq 0 \)

$$
L^{-1} \tilde{\mu}_t \leq \mu_t \leq L \tilde{\mu}_t \quad (5.5)
$$
on all of \( M_t \). This is possible since by Proposition 5.3

$$
0 < \inf_{|t|^{1/2} < |z| < l^{-1}} (-|z|^2 \log|t|/\mu_t(z)) \leq \sup_{|t|^{1/2} < |z| < l^{-1}} (-|z|^2 \log|t|/\mu_t(z)) < +\infty.
$$

Let \( \{\lambda_n(t)\}_{n=0}^\infty \) be the eigenvalues for \( \mu_t \) and \( \{\tilde{\lambda}_n(t)\}_{n=0}^\infty \) those for \( \tilde{\mu}_t \). Then we have the following theorem.
THEOREM 5.6 (E. B. Davies, [D] Theorem 3). Under the assumption equation 5.5

$$L^{-4/5}\lambda_n(t) \leq \lambda_n(t) \leq L^{4/5}\lambda_n(t)$$

holds for all $n \geq 0$ and $t \neq 0$.

Proof of Theorem B, part (ii). By monotonicity it suffices to show that the Dirichlet and Neumann spectra of a subdomain become continuous as $t \to 0$ while the spectrum on the complement is controlled. By Theorem 5.6 we may equivalently consider the eigenvalue problem for $\mu$. But for $\mu$, the domain $|z| < \frac{1}{2}l^{-1}$ in local coordinates about the node is isometric to a flat cylinder of length $\sim l$ and circumference $2\pi l^{-1}$. The Dirichlet eigenvalues for the cylinder are

$$\lambda_{m,n} = \left(\frac{m\pi}{l}\right)^2 + (nl)^2, \quad m = 1, 2, \ldots; n = 0, 1, 2, \ldots$$

and the Neumann eigenvalues $\{\mu_{m,n}\}$ are the same, where we allow $m = 0$. As $l \to \infty$, $\{\lambda_{m,n}\}$ and $\{\mu_{m,n}\}$ clearly become dense on the entire interval $[0, +\infty)$. On the complement of the region $\{z| |z| < l^{-1}\}$, one can bound the isoperimetric constant away from zero, as in the proof of part (i), and by the results of Section 3, the Dirichlet and Neumann spectra converge. Finally, in the region $\{z| \frac{1}{2}l^{-1} < |z| < l^{-1}\}$ the annulus is collapsing to a circle. It is easy to see that Cheeger's constant diverges, and since it is a lower bound for the entire Dirichlet spectrum, the latter also diverges. The circumference of the annulus remains bounded away from zero. By decomposing into phases, we see that for small $l$ there are only finitely many Neumann eigenvalues in any interval. This follows from the divergence of the Dirichlet spectrum, and the fact that, for each phase, Neumann eigenvalues can be bounded below by Dirichlet eigenvalues after shifting the index by two. (See [We].) Now the proof of part (ii) follows by monotonicity.

Remark 5.7. Heuristically, the fact that for small $|t|$ we have an embedded cylinder which is close to being flat means that in the limit we get the continuous spectrum of the real line, i.e., $[0, +\infty)$. This is in contrast to the case of the hyperbolic metric where this type of argument can be made rigorous; an embedded hyperbolic cylinder produces continuous spectrum only in the interval $[1/4, +\infty)$. (See [Ji].)

Remark 5.8. The long, thin cylinder may be understood geometrically—for the nonseparating case, the Jacobian variety $J(M_t)$ becomes a noncompact torus as $t \to 0$. Furthermore, from the embedding $M_t \to J(M_t)$, it can be seen (see [W]) that the pinching annulus wraps around that part of the torus which becomes unbounded. As $t \to 0$, we therefore produce a long, thin cylinder.

COROLLARY 5.9. Let $\lambda_1(t)$ denote the first nonzero eigenvalue for the degenerating family $\mathcal{M}$ with Bergman metrics. Then $\lambda_1 \to 0$ for both the separating and nonseparating cases.
Proof. For the nonseparating case this follows from monotonicity and the fact that infinitely many eigenvalues converge to zero for the long, thin cylinder. The limiting spectrum in the separating case is the union of the two spectra, and hence contains two zero eigenvalues. Since $\lambda_1(t)$ is the second eigenvalue in $\text{Spec}(\Delta_t)$, $\lambda_1(t)$ must go to zero.

Remark 5.10. We may guess at how fast $\lambda_1 \to 0$ in the separating case. The method used suggests that $\lambda_1(t)$ should behave like the Dirichlet eigenvalue $\lambda_1(\varepsilon)$ for the complement of the set $\{q | |z(q)| < \varepsilon\}$ on one of the two surfaces $M_i$. By a result of Ozawa [O]

$$\lambda_1(\varepsilon) = -\frac{2\pi}{\text{Area}(M_i)}(\log \varepsilon)^{-1} + O((\log \varepsilon)^{-2}).$$

Since we may take $\varepsilon \sim |t|^{1/2}$, we expect $\lambda_1(t)$ to be bounded above by a multiple of $(\log |t|)^{-1}$. Note that this is the behavior of $\lambda_1$ for the hyperbolic metric; however, in the nonseparating case $\lambda_1$ is bounded away from zero for the hyperbolic metric, in contrast to Corollary 5.9. (See [SWY].)

6. Proof of Theorem C. The Bergman metric of Section 5 degenerates to a smooth metric. In this section we study the admissible metrics introduced by Arakelov [A]; these degenerate to cone metrics. It will be convenient however to have a different description of cone metrics on surfaces. We have the following simple lemma.

**Lemma 6.1.** Let $g$ be a metric on a two-dimensional manifold $M \setminus \{p\}$ such that, in local coordinates $x$ centered at $p$, $g = \|x\|^{-2a}x^* ds^2$, where $ds^2$ is the standard Euclidean metric and $a$ is some number $0 \leq a < 1$. Then $g$ is a cone metric on $M$.

**Proof.** Let $(r, \theta)$ be polar coordinates associated to $x$, $(\tau, \phi)$ coordinates on the cone $C(S^1_a)$, where $S^1_a$ is the circle of radius $1 - a$ with the standard metric $\bar{g}(\phi) = (1 - a)^2 d\phi^2$. Consider the map

$$(r, \theta) \mapsto \left( \frac{r^{1-a}}{1-a}, \theta \right).$$

This defines a smooth diffeomorphism from a deleted neighborhood of $p$ to $C(S^1_a)$ and the standard metric $ds^2_C = d\tau^2 + \tau^2 \bar{g}(\phi)$ pulls back to $g$. Hence, by Definition 2.1, $g$ is a cone metric.

As in Section 5, we fix a compact Riemann surface $M$ of genus $g > 0$ and let $\mu$ denote the Bergman metric.

**Definition 6.2.** The Arakelov-Green's function on $M$, denoted $G(z, w)$, is characterized by the following

(i) $G(z, w)$ has a zero of order one on the diagonal in $M \times M$;

(ii) $G(z, w) = G(w, z)$;
(iii) for $z \neq w$, $\partial_z \partial_{\overline{z}} \log G(z, w) = -\frac{2}{\pi} \mu(z);
(iv) \int_M \log G(z, w) \mu(z) |dz|^2 = 0.$

**Definition 6.3.** A metric $ds^2 = \rho(z) |dz|^2$ on $M$ is called *admissible* if its Ricci form is proportional to the Kähler form of the Bergman metric; i.e.,

$$\partial_z \partial_{\overline{z}} \log \rho(z) = 2\pi(g - 1) \mu(z).$$

**Remark 6.4.** The exact multiple follows from Gauss-Bonnet and the fact that $\int_M \mu(z) |dz|^2 = 1$. Note that an admissible metric always has negative curvature for $g \geq 2$ and that any two admissible metrics are proportional. For tori the metric is flat.

**Definition 6.5.** (i) Set

$$G(z, w) \lim_{|z - w| \to 0} [z - w]^{-2},$$

Then $\rho$ is an admissible metric and defines the *Arakelov metric* (see [A]).

(ii) The normalized admissible metric $\tilde{\rho}$ is the admissible metric with unit area. By Remark 6.4, $\tilde{\rho}(z) = \rho(z)/\text{Area}(M, \rho)$, where $\rho$ is the Arakelov metric.

Now let $\mathcal{M}$ be an analytic family as described in Section 5. The Arakelov metrics $\rho_t$ form a smooth family for $t \neq 0$, and their behavior as $t \to 0$ has been studied in [W]. We are interested in $\text{Spec}(\Delta_\rho_t)$.

**Proposition 6.6.**

(i) Let $\mathcal{M}$ be degenerating to a separating node with $j < g/2$. (see the beginning of Section 5 for notation.) Then

$$\tilde{\rho}_t(z) \to \rho_1(z) G_1(z, p_1)^{-4j/g}$$

uniformly on compact subsets of $M_1 \setminus \{p\}$. Here, $\rho_1$ is an admissible metric for $M_1$, and $G_1$ is the Arakelov-Green's function for $M_1$. Moreover, $\tilde{\rho}_t$ vanishes to order $|t|^{2(1 - 2j/g)}$ uniformly on compact subsets of $M_2 \setminus \{p_2\}$.

(ii) Let $\mathcal{M}$ be degenerating to a nonseparating node. Then

$$\tilde{\rho}_t(z) \to \rho(z)(G(z, a) G(z, b))^{-2/g}$$

uniformly on compact subsets of $M_0 \setminus \{p\}$. As above, $\rho$ is an admissible metric for $M$, and $a$ and $b$ are as in Section 5.

**Remark 6.7.** The form of the limiting metrics (note that they are quasi-isometric to cone metrics by property (i) in Def. 6.2, Lemma 6.1, and the assumption on $j$) follows from the results in [W]. However, the asymptotic behavior for the Arakelov metric $\rho_t$ studied there only gave pointwise results away from the node. To determine the limiting behavior of $\tilde{\rho}_t = \rho_t/\text{Area}(\rho_t)$, we need estimates on the area as well.
Remark 6.8. The limiting metrics are “admissible cone metrics”; that is, their curvature is a multiple of the Bergman metric. In the cases considered above, the limiting metrics have nonpositive curvature bounded from below. (The curvature is negative if the limiting surfaces are not tori.)

To obtain the proposition we must control the behavior of the metric in the pinching region better than in [W]. To do this, we use Fay’s expression for admissible metrics; let $\mathcal{H}[f]$ denote the theta function with characteristic $f$ associated to $J(M)$. Choose $f$ to be an odd, nonsingular element of the theta divisor in $J(M)$.

**Proposition 6.9.** Let $\rho(z) = |H_f(z)|^2 \Psi(z)$, where

$$H_f(z) = \sum_{j=1}^{g} \frac{\partial}{\partial Z_j} \mathcal{H}[f](0) \omega_j(z),$$

$$\Psi(z) = \exp\left\{ \frac{4}{g} (g - 1) \pi \sum_{i,j=1}^{g} \text{Im} \, \Omega_{ij}^{-1} \text{Im} \left( \int_{z_0}^{z} \omega_i - k_i \right) \text{Im} \left( \int_{z_0}^{z} \omega_j - k_j \right) \right\},$$

$$- \frac{2}{g} \sum_{j=1}^{g} \text{Re} \int_{A_j} \omega_j(\xi) \log \frac{\mathcal{H}[f](i\xi, \bar{\omega})^2}{H_f(\xi)} \frac{d\xi}{2\pi(g-1)/(z)}.$$

$z_0$ is an arbitrary point of $M$, and $k$ is a point in $J(M)$ depending upon $z_0$. Then $ds^2 = \rho(z) |dz|^2$ is an admissible metric.

**Sketch of proof.** (See [F2] for details.) The zeros of $H_f(z)$, all of multiplicity two, coincide with the zeros of $\mathcal{H}[f](i\xi, \bar{\omega})$; so $\rho(z)|dz|^2$ is nonsingular. The factors of automorphy of $\mathcal{H}$ cancel those for the first term in the exponential; so $\Psi(z)$ is indeed a single-valued function on $M$ depending, however, on the choice of homology basis. Now by a simple computation

$$\partial_z \bar{\partial}_z \log \rho(z) = \partial_z \bar{\partial}_z \left\{ \frac{4}{g} (g - 1) \pi \sum_{i,j=1}^{g} \text{Im} \, \Omega_{ij}^{-1} \text{Im} \left( \int_{z_0}^{z} \omega_i - k_i \right) \text{Im} \left( \int_{z_0}^{z} \omega_j - k_j \right) \right\}$$

$$= 2\pi(g - 1) \mu(z).$$

Using this expression, we shall prove Proposition 6.6. For brevity we shall only prove part (i); part (ii) follows similarly.

**Lemma 6.10.** Let $\mathcal{M}$ be a degenerating family as in part (i) of Proposition 6.6. Then we may choose $f_t$ analytic in $t$ such that

$$H_{f_t}(z) \to \alpha H_{f_0}(z)$$

uniformly on compact subsets of $M_1 \setminus \{p_1\}$, where $\alpha$ is some constant and $f_0$ is a nonsingular odd element of the theta divisor in $J(M_1)$. Moreover, if $G_2$ denotes the
Arakelov-Green’s function on $M_2$, then $|t|^{-1} G_2(z, p_2)^2 H_f$ is uniformly bounded for $t \in D \setminus \{0\}$ and $z \in M_2 \cap M_1$.

**Proof.** This is a simple consequence of the degeneration formulas in [F1]. The Jacobian variety degenerates to a product torus, and the theta divisor over the zero fiber

$$\Theta_0 = \Theta_1 \times J(M_1) \cup J(M_2) \times \Theta_2.$$  

(See [W].) Choose $f_i$ such that $\lim_{t \to 0} f_i$ is in $\Theta_1 \times J(M_2)$ with $f_0$ in the first factor. Then for $i \leq j$

$$\frac{\partial}{\partial z_i} g(f_i) \to g_2, \quad \frac{\partial}{\partial z_i} g_1(f_0)$$

and vanishes otherwise. Since the normalized abelian differentials are chosen such that $\omega_t(z, t)$ converges uniformly away from the node to the abelian differentials of the compact surface, and $\omega_t(z, t) \to 0$ for $i \leq j$ and $z \in M_2 \setminus \{p_2\}$, we have the first part of the lemma with $\alpha = g_2$ evaluated on the second factor of $\lim_{t \to 0} f_i$. The second part follows from the fact that $H_f \to 0$ to order $t$ on compact subsets of $M_2 \setminus \{p_2\}$ and that near the node $H_f(z) \sim t dz/z^2$. (See Appendix A of [W].) Since $G_2(z, p_2)^2 \sim |z|^2$ near the node, the result follows.

**Lemma 6.11.** Let $R$ be any region in $M_1 \setminus \{p\}$ and $\gamma$ any smooth family of curves in $M_1 \cap M$. Then there exists a positive constant $C$ independent of $t$ such that

(i) $\int_{R \cap M_1} |H_{f_t}(z) - \alpha H_{f_0}(z)|^2 |dz|^2 \leq -C|t| \log |t|,$

(ii) $\int_{\gamma} |H_{f_t}(z) - \alpha H_{f_0}(z)| |dz| \leq -C|t|^{1/2} \log |t|.$

**Proof.** See [W], Propositions A.1 and A.4.

**Lemma 6.12.** Given $f_i$ as in Lemma 6.10, there exists a bounded function $\Psi_0$ on $M_0 \setminus \{p\}$ such that

$$\Psi_t(z)|t|^{-2j/g} G_1(z, p_1)^{4j/g} \to \Psi_0(z) \quad \text{uniformly for } z \in M_1 \cap M_1;$$

$$\Psi_t(z)|t|^{2j/g} G_2(z, p_2)^{-4j/g} \to \Psi_0(z) \quad \text{uniformly for } z \in M_2 \cap M.$$

Moreover, $\Psi_0(z)|H_{f_t}(z)|^2$ is an admissible metric on $M_1$.

**Proof.** This may be proven by applying the degeneration formulas in [F1] to the explicit expression. We shall not go through the details since the answer was essentially obtained in [W]. Let us note only that uniformity on all of $M_0$ follows
from the fact that the $z$ dependence of $\Psi_t$ is in terms of abelian integrals, and we may again apply Proposition A.1 of [W] to see that the limits are uniform.

**Proof of Proposition 6.6, part (i).** Let $C(t) = |t|^{-2j/g}$ and set

$$\rho_t(z) = C(t)\Psi_t(z)|H_{f_t}(z)|^2.$$  

By Proposition 6.9, $\rho_t$ is a smooth family of admissible metrics for $t \neq 0$. Furthermore,

$$\text{Area}(M_t, \rho_t) = A_t = \int_{M_t} C(t)\Psi_t(z)|H_{f_t}(z)|^2 \, |dz|^2 = \sum_{M_i \cap M_t} \cdots + \sum_{M_i \cap M_t} \cdots.$$  

Treating first the second term,

$$\int_{M_t \cap M_t} C(t)\Psi_t|H_{f_t}|^2 \, |dz|^2 = \int_{M_t \cap M_t} |t|^{2j/g} G_2(z, p_2)^{-4j/g} |\Psi_t| \, |t|^{-2} G_2(z, p_2)^4 |H_{f_t}|^2 \, |dz|^2.$$  

By Lemmas 6.10 and 6.12 the integrand is dominated (in local coordinates) by a multiple of $|t|^{2(1-2j/g)} G_2(z, p_2)^{-4(1-j/g)}$. For $z$ near $p_2$, we estimate

$$\int_{|t|^{1/2} < |z| < 1} |t|^{2(1-2j/g)} G_2(z, p_2)^{-4(1-j/g)} \leq \text{const.} \, |t|^{2(1-2j/g)} \int_{|t|^{1/2}}^1 dr r^{-3+4j/g} \leq \text{const.} \, |t|^{1-2j/g}.$$  

Since we assume $j/g < 1/2$, we conclude from the above that

$$\lim_{t \to 0} \int_{M_t \cap M_t} C(t)\Psi_t|H_{f_t}|^2 \, |dz|^2 = 0.$$  

For the first term let $\tilde{\Psi}_0(z) = \Psi_0(z)G_1(z, p_1)^{-4j/g}$. Then

$$\int_{M_t \cap M_t} C(t)\Psi_t|H_{f_t}|^2 \, |dz|^2 = \alpha^2 \int_{M_t \cap M_t} \tilde{\Psi}_0|H_{f_0}|^2 \, |dz|^2 \quad (6.13)$$

$$+ \int_{M_t \cap M_t} \{C(t)\Psi_t|H_{f_t}|^2 - \alpha^2 \tilde{\Psi}_0|H_{f_0}(z)|^2 \} \, |dz|^2.$$
By uniform convergence, for any $\delta > 0$ the second term in equation 6.13 may be bounded by

\[
\int_{M_t \cap M_t} |(C(t)\Psi_t - \tilde{\Psi}_0)|H_{f_t}|^2 + \tilde{\Psi}_0(|H_{f_t}|^2 - \alpha^2|H_{f_0}|^2)|dz|^2
\leq \delta \int_{M_t \cap M_t} \alpha^2|H_{f_0}|^2|dz|^2 + \delta \int_{M_t \cap M_t} |H_{f_t} - \alpha H_{f_0}|^2|dz|^2
+ \sup_{z \in M_t \cap M_t} (\tilde{\Psi}_0(z)) \int_{M_t \cap M_t} |H_{f_t} - \alpha H_{f_0}|^2|dz|^2
\]

for sufficiently small $|t|$. The last two terms $\to 0$ as $t \to 0$ by Lemma 6.11 and the assumption $j < g/2$, and since $\delta$ was arbitrary, we conclude that the second term on the right-hand side of equation 6.13 vanishes as $t \to 0$. We have shown

\[
\lim_{t \to 0} \left\{ \int_{M_t \cap M_t} \alpha^2|H_{f_0}|^2|dz|^2 \right\} = 0.
\]

Hence, the normalized admissible metric $\rho_t = \rho_t/A_t$ converges as in Proposition 6.6, part (i), completing the proof.

**Proposition 6.14.** Let $\mathcal{M}$ be as in Proposition 6.6. Then there exists a constant $c$ depending only on $\mathcal{M}$ such that for all $t \in D \setminus \{0\}$

\[
\mathscr{F}(M_t) \geq c > 0.
\]

**Remark 6.15.** Note that we make no assumption on $\mathcal{M}$. In particular, $\mathscr{F}(M_t)$ is bounded away from zero even in the separating case. (Compare with Remark 3.12.) The reason for this is that by Proposition 6.6 one entire side of the degenerating surface is collapsing, and a separating curve in the pinching annulus has squared length comparable to the area of this collapsing piece. As in Section 5, we also note that Proposition 6.6 and 6.14, combined with the arguments in Section 4 immediately prove Theorem C, part (i).

**Proof of Proposition 6.14.** Again, we only consider the separating case. Let $\gamma_t$ be a smooth family of closed curves in $M_t$. Suppose that $\gamma_t \subset M_t \cap M_t$. Then we estimate

\[
|L(\gamma_t, \rho_t) - L(\gamma_t, \rho_0)| = \int_0^1 ds|\dot{\gamma}_t(s)||\sqrt{\rho_t} - \sqrt{\rho_0}|
= \int_0^1 ds|\dot{\gamma}_t|||H_{f_t}|(C(t)\Psi_t)^{1/2} - \alpha|H_{f_0}|(\tilde{\Psi}_0)^{1/2}|
\]
The first term \( \rightarrow 0 \) as \( t \rightarrow 0 \) by the uniform convergence of \( C(t)\Psi_t \). The second term is bounded by a multiple of \(-|t|^{1/2-j/\theta} \log |t|\) by Lemmas 6.11 and 6.12. Area estimates follow as in the proof of Proposition 6.6 above. Thus, restricting \( \gamma \) in \( M_1 \), the isoperimetric quotient may be bounded by that of the metric \( \rho_0 \), which by the general arguments of Section 2 is bounded away from zero. Notice that, by Remark 6.4 and Theorem 2.7, we restrict to homotopically nontrivial loops. For \( \gamma \) restricted to a compact subset of \( M_2 \setminus \{p_j\} \), the metric \( \bar{\rho} \) may be scaled by \(|t|^{-2(1-j/\theta)}\) to converge to a smooth metric whose isoperimetric constant is bounded away from zero, and hence the same for \( \bar{\rho} \) by the scale invariance of \( \mathcal{F}(M) \). The last case to consider is when \( \gamma \) is in the pinching annulus in \( M_2 \). Again, by Remark 6.4 and the arguments of Section 2, we may restrict ourselves to rotationally symmetric curves. One has from Lemmas 6.10 and 6.12 that, in \( M_2 \setminus M_\theta \), \( \bar{\rho} \) is uniformly quasi-isometric to \(|t|^{2(1-j/\theta)}|z|^{-4(1-j/\theta)}\), where \( z \) is the local coordinate about the node. It is easy to see that the isoperimetric quotient is bounded away from zero for this metric as well. This completes the proof of Proposition 6.14.

It remains to prove Theorem C in the case of degeneration to a separating node where both surfaces have the same genus \( j = g/2 \).

**Lemma 6.16.** Let \( \gamma \) be degenerating to a separating node where the surfaces \( M_1 \), \( M_2 \) both have genus \( g/2 \). Let \( \rho_t \) denote the Arakelov metric on \( M_t \). Then

\[
|t|^{-1/2} \rho_t(z) \rightarrow \rho(z)(G(z, p_i))^{-2}
\]

uniformly on compact subsets of \( M_1 \setminus \{p_i\} \). Here, \( \rho_i \) is the Arakelov metric of \( M_i \).

**Proof.** This is just equation 8.1 of [W].

**Lemma 6.17.** Let \( A_t = \text{Area}(M_t, \rho_t) \). Then

\[
A_t = O(|t|^{1/2} \log |t|).
\]

**Proof.** That this naive guess is correct follows from the explicit expression, Proposition 6.9, and Lemmas 6.10, 6.11 and 6.12. (The proofs of these did not depend on \( j < g/2 \).) Note especially the “extra” factor of \(|t|^{1/2}\) in Lemma 6.11, part (i).

**Proof of Theorem C, part (ii).** Fix a \( \delta > 0 \). Then for small \( |t| \) we construct a family of interpolating metrics \( \bar{\mu}_t \) as in Section 5, satisfying

1. \( \bar{\mu}_t = \bar{\rho} \) on the complement of the pinching annulus \( \{z| |z| < \delta\} \) in local coordinates about the node;
2. \( \bar{\mu}_t(z) = (\log |t|)^{-1} |z|^{-2} \) for \( |z| < \delta/2 \);
3. \( \sup_{(t/2 < |z| < \delta)} (\log |t| \bar{\mu}_t(z)) \) is bounded independently of \( t \).
Then by Lemmas 6.16 and 6.17 (and their proofs) we can find an \( L > 0 \), independent of \( t \), such that for \( t \neq 0 \)

\[ L^{-1} \tilde{\mu}_t \leq \tilde{\mu}_t \leq L \tilde{\mu}_t, \]

on all of \( M_t \). By Theorem 5.6, \( \text{Spec}(\Delta_{\bar{g}_t}) \) is bounded above and below by \( \text{Spec}(\Delta_{\bar{g}}) \). The region \( \{ |z| < \delta/2 \} \) is a flat cylinder with respect to \( \tilde{\mu}_t \), and its spectrum becomes dense in \( [0, +\infty) \) as \( t \to 0 \). (See Section 5.) On the complement of the region, we can, by assumption 1 above and Lemma 6.16, rescale by a factor of \(-\log|t|\) to obtain a smoothly converging metric with converging spectrum. Hence, the Dirichlet and Neumann spectra for \( \tilde{\mu}_t \) diverge on this piece. The proof now follows from monotonicity.

REFERENCES


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