Affine Automorphisms of Properly Convex Domains

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A *cone* Ω is any subset of an affine space which is invariant under a one parameter group of positive homotheties that have the extreme point of the cone as a fixed point. We denote the group of affine automorphisms which preserve Ω by $\mathrm{Aut}(\Omega)$. When there is a discrete subgroup $G \subset \mathrm{Aut}(\Omega)$ which acts properly on Ω, the quotient $G\backslash\Omega$ is an affine manifold, and Ω is a covering space.

A convex cone is said to be *properly convex* or *sharp* if it contains no complete affine line. Any convex cone Ω in an n-dimensional affine space is isomorphic to a product $\mathbb{R}^k \times \Omega'$ where $k \leq n$ and Ω' is a sharp convex cone [4].

If $\mathrm{Aut}(\Omega)$ acts transitively, Ω is said to be *homogeneous.* Of course quotients by a subgroup acting transitively are not particularly interesting, so we consider a slight weakening of this notion: Ω is said to be *quasi-homogeneous* if there is a subgroup $G \subset \mathrm{Aut}(\Omega)$ and a compact $K \subset \Omega$ such that $GK = \Omega$. G is said to *act syndetically* on Ω. Other terminology (particulary in [6]) for this is G *sweeps* Ω. The quotient $G\backslash\Omega$ is then compact but not necessarily Hausdorff.

An open domain Ω in affine space is said to be *divisible* if it is quasi-homogeneous and if G acts properly on Ω when G is given the discrete topology. This condition ensures that the quotient $G\backslash\Omega$ is compact *and Hausdorff.*

The main goal of this paper is the proof of the following theorem, which was obtained in 1970 by Jaques Vey [6]

Theorem 1. *Every properly convex divisible domain is a cone.*

1 Some Invariants on Convex Cones

Let Ω be a sharp convex cone in a finite dimensional affine space. By choosing the cone point of Ω to be the origin, we can consider Ω to be a subset of a finite dimen-

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sional vector space V. Denote the dual space V^*. We define the *dual cone*, Ω^* by

$$\Omega^* = \{\psi \in V^* \, | \psi(x) > 0 \; \forall x \in \overline{\Omega} - \{0\}\}$$

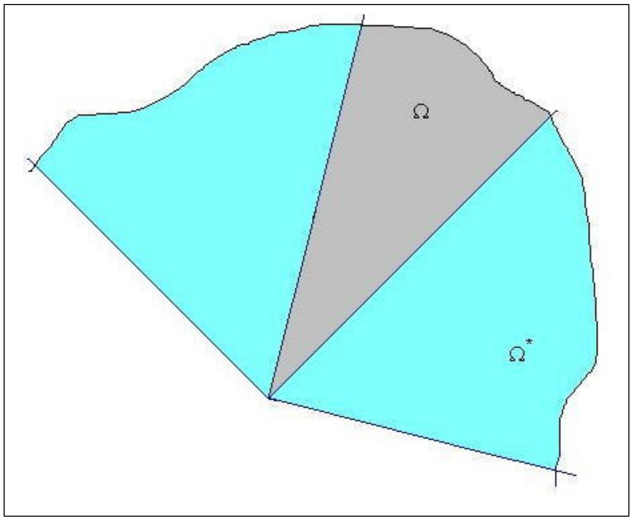

Figure 1: A Cone and its dual

The dual cone is nonempty provided Ω is sharp, and it is a sharp convex cone if Ω has nonempty interior. This duality will be useful in the next section.

The *Hilbert Metric* for any sharp convex cone can be defined as follows: Let p, q be distinct points of Ω. The line $\overleftrightarrow{pq}$ intersects the boundary in 2 points a, b. Note that Ω is necessarily unbounded, and can be considered to be contained in an affine patch of projective space, so a or b could possibly be points at infinity. Let $[a, p, q, b] = \frac{(q-a)(p-b)}{(p-a)(q-b)}$ be the cross ratio of these 4 points. The hilbert metric $d_H : \Omega \times \Omega \to \mathbb{R}^+$ is

$$d_H(p, q) = \log |[a, p, q, b]|$$

We need to impose the final condition $d_H(p, p) = 0$ to make d_H a metric on Ω. Since the cross ratio is invariant under projective transformations [2], the Hilbert metric is invariant under $\mathrm{Aut}(\Omega)$. It is worth nothing that there are 6 permutations of the cross ratio and this choice of the cross ratio is necessary to ensure that d_H is positive. For proof that d_H is a metric see [1]

2 Necessary Lemmas

The following lemma is a known fact from geometry.

Lemma 1. Let X be any compact metric space. Then any distance non-increasing homeomorphism $T : X \to X$ is an isometry

Proof. Choose $x_0, y_0 \in X$, and define sequences $x_i = T^{-1}x_{i-1}$ and $y_i = T^{-1}y_{i-1}$. Since X is compact, we can choose a subsequence x_{i_k} which converges to $x \in X$, then from the sequence y_{i_k} choose a subsequence $y_{i_{k_j}}$ which converges to some $y \in X$. So without any loss of generality, we can assume that $x_i \to x$ and $y_i \to y$.

Since T is distance non-increasing, T^{-1} is distance increasing, and hence

$$d(x_0, x_1) \leq d(x_1, x_2) \leq ... \leq d(x_i, x_{i+1}) \leq ...$$

and similarly for $\{y_i\}$. Let $\epsilon > 0$. Then there is N such that for all $n, m > N$, $d(x_n, x_m) < \epsilon$ and $d(y_n, y_m) < \epsilon$. So

$$d(x_0, x_1) \leq d(x_1, x_2) \leq ... \leq d(x_n, x_{n+1}) < \epsilon$$

and similarly for $\{y_i\}$. In particular we have $d(x_0, x_1) < \epsilon$ and $d(y_0, y_1) < \epsilon$. Now consider the inequalities:

$$\begin{aligned} d(x_0, y_0) &\leq d(T^{-1}x_0, T^{-1}y_0) = d(x_1, y_1) \\ &\leq d(x_0, x_1) + d(x_0, y_1) && \textit{Triangle inequality} \\ &\leq d(x_0, x_1) + d(x_0, y_0) + d(y_0, y_1) && \textit{Triangle inequality} \\ &\leq d(x_0, y_0) + 2\epsilon \end{aligned}$$

hence we have $d(x_0, y_0) \leq d(x_1, y_1) \leq d(x_0, y_0) + 2\epsilon$ and in fact $d(x_0, y_0) = d(x_1, y_1)$, so T is an isometry. □

The proof of Vey's result will rely on the following:

Lemma 2. Let Ω be a sharp divisible convex cone, and G a subgroup dividing Ω. Let H be a supporting hyperplane of Ω which is stable under the action of G. Then there is a one dimensional supplementary subspace L which is also G-stable, so $V = H \oplus L$.

Any hyperplane $H \subset V$ determines a 1-dimensional subspace in V^*, specifically those linear functionals whose kernel is H. Similarly any line in V determines a hyperplane in V^*. Operating under the assumptions that Ω is sharp and has non-empty interior, Ω and Ω^* are both sharp convex cones with nonempty interior, and hence Lemma 2 is then equivalent to the following:

Lemma 3. Let Ω be a sharp divisible convex cone, and G a subgroup dividing Ω. Let L be a 1 dimensional subspace of V which is G-stable and $L \cap \overline{\Omega} \neq 0$. Then there is a hyperplane H supplementary to L which is also stable under G, so $V = L \oplus H$.

Proof of Lemma 3. Let L be a line which is invariant under G and intersects nontrivially with $\overline{\Omega}$. For each $x \in \Omega$, let $\Omega_x = \Omega \cap (x + L)$. Since Ω is sharp, Ω_x is a ray, with endpoint on the boundary of Ω. Let $s : \Omega \to \partial\Omega$ take x to the endpoint of Ω_x.

Observe that s commutes with G: By definition $x - s(x) \in L$, and L is G-stable so $gx - gs(x) \in L$. Of course g preserves the boundary of Ω, so $gs(x) \in \partial\Omega$, hence $gs(x) = s(gx)$.

For each $t \in \mathbb{R}$ let

$$c_t(x) = s(x) + e^t(x - s(x))$$

Claim: This one parameter group of homeomorphisms of Ω is distance non-increasing with respect to the Hilbert metric, i.e. $d_H(c_t(x), c_t(y)) \leq d_H(x, y)$

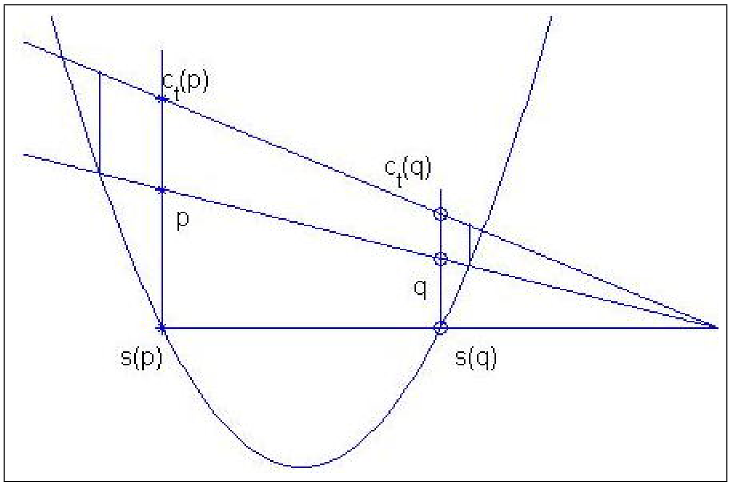

Figure 2: c_t is a distance non-increasing homeomorphism

To see this, choose $p, q \in \Omega$ (See figure 2). Clearly $c_t(p) - p$ and $c_t(q) - q$ are both in L. Let a and b denote the intersections of $\overleftrightarrow{pq}$ with $\partial\Omega$, and c and d denote the intersections of $\overleftrightarrow{c_t(p)\ c_t(q)}$ with $\partial\Omega$. Now let $\hat{a}$ and $\hat{b}$ denote the intersections of $a + L$ and $b + L$ respectively with $\overleftrightarrow{c_t(p)\ c_t(q)}$. By convexity of Ω, $\overline{\hat{a}\hat{b}} \subset \overline{cd}$, hence $[c, c_t(p), c_t(q), d] \leq [\hat{a}, c_t(p), c_t(q), \hat{b}] = [a, p, q, b]$. The last equality is due to the invariance of the cross ratio under projective transformations (in particular this is a perspectivity). Since the cross ratio is non-increasing, it follows that d_H is non-increasing as well.

Since s commutes with G, we can pass the map c_t down to a map C_t on the quotient. $C_t : G\backslash\Omega \to G\backslash\Omega$ given by $C_t(Gx) = Gc_t(x)$, and this map is distance non-increasing with respect to the induced metric on the quotient. Note that the quotient is only a metric space because G **divides** Ω. So C_t is a distance non-increasing homeomorphism of $G\backslash\Omega$. So C_t satisfies the hypotheses of Lemma 1 , hence C_t is an isometry. This now implies that c_t *is a local isometry on* Ω.

Note: Figure 2 is a bit misleading here since the domain pictured is not a cone, and c_t is actually strictly decreasing for this domain. Figure 3 below, shows c_t on a cone, and in this picture we see that c_t is actually an isometry.

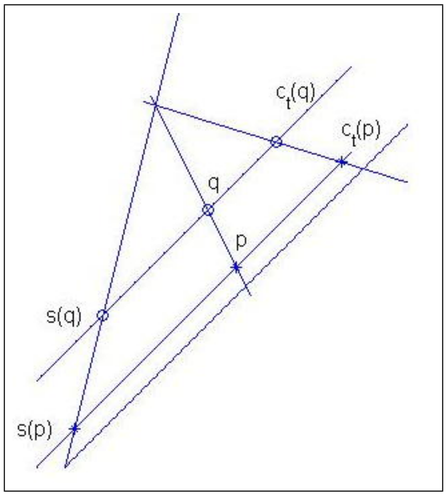

Figure 3: c_t is an isometry of the cone

We now claim that the set $s(\Omega)$ is convex: Suppose by way of contradiction that there are $x, y \in \Omega$ such that the line segment $\overline{s(x)\ s(y)}$ is not contained in $s(\Omega) \subset \partial\Omega$. Let $p \in \overline{s(x)\ s(y)}$ be such that $p \notin s(\Omega)$. Let N be a neighborhood of p on which c_t is an isometry, and choose $q \in (\overline{s(x)\ s(y)}) \cap N$. Choose $t < 0$ such that both $c_t(p)$ and $c_t(q)$ are contained in N. Let a, b be the intersections of $\overleftrightarrow{c_t(p)\ c_t(q)}$ with $\partial\Omega$. Let $\hat{a} = (a + L) \cap \overleftrightarrow{pq}$ and $\hat{b} = (b + L) \cap \overleftrightarrow{pq}$. Clearly now $\hat{a}$ is in $\overline{s(x)\ s(y)}$, but not equal to $s(x)$: if it were equal to $s(x)$ then $a = s(x)$ and $t = 0$. Similarly $\hat{b}$ is on $\overline{s(x)\ s(y)}$ and not equal to $s(y)$. Now we have that $\overline{\hat{a}\ \hat{b}} \subsetneq \overline{s(x)\ s(y)}$, hence $[a, c_t(p), c_t(q), b] = [\hat{a}, p, q, \hat{b}] < [s(x), p, q, s(y)]$ and $d_H(c_t(p), c_t(q)) < d_H(p, q)$, contradicting that c_t is an isometry on N.

Now $s(\Omega)$ is a convex set contained in the boundary of Ω, so it generates a hyperplane H. Now $V = L + H$ and this sum is actually a direct sum: for any $x \in \Omega$, $x - s(x) = l$ for some $l \in L$, so $x = l + s(x)$ and hence $V = L \oplus H$.

Since s commutes with G, H is G-stable which completes the proof. $\square$

Lemma 3 is actually a weaker version than what was proved by Vey in [6], but

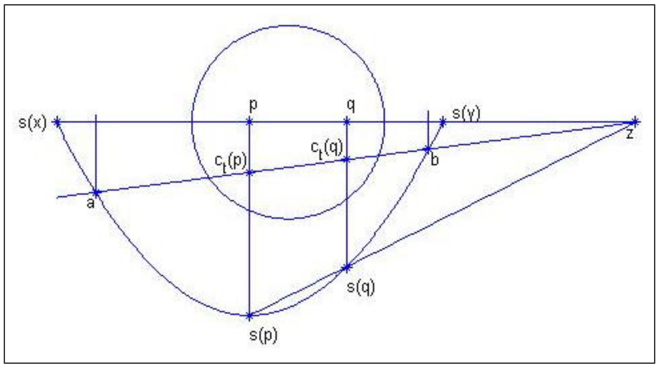

Figure 4: Proving $s(\Omega)$ is convex

it will be all that is necessary for the proof of theorem 1. In his paper, Vey actually proved:

Theorem 2. *Let Ω be a sharp divisible convex cone, and G a subgroup dividing Ω. Then the action of G is semi-simple.*

3 Proof of Vey's Theorem

Theorem 1: Every properly convex divisible domain is a cone.

Proof. Let Ω be a properly convex divisible domain in an n dimensional affine space. Construct the cone over Ω in $\mathbb{R}^{n+1}$:

$$C(\Omega) = \{(x,t) \mid x \in \mathbb{R}^n, t > 0 \in \mathbb{R}, \tfrac{x}{t} \in \Omega\}.$$

$C(\Omega)$ is now a sharp convex cone, and the plane $L_1 = \{(x,1) \in \mathbb{R}^{n+1}\}$ can be identified with the original affine space E containing Ω. Further, Ω is identified with $\Omega_1 = \{(x,1) | x \in \Omega\}$. The group G dividing Ω can be identified with a group of automorphisms preserving $L_1 \cap \Omega_1$, and we obtain a homomorphism $\Phi : G \to \mathrm{Aut}(C(\Omega))$ given by

$$Ax + b \to \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

Let $\tilde{G}$ be the subgroup of $\mathrm{Aut}(C(\Omega))$ generated by $\Phi(G)$ and homotheties $\Gamma = \{2^k : k \in \mathbb{Z}\}$, so $\tilde{G} =< \Phi(G), \Gamma >$. *Note that $\tilde{G}$ preserves the supporting hyperplane L_0.*

$\tilde{G}$ now divides $C(\Omega)$: Let $K \subset \Omega$ $(= \Omega_1)$ be compact such that $GK = \Omega_1$. Let $\tilde{K} = K \times [1,2]$. Now $\tilde{G}\tilde{K} = C(\Omega)$, so $C(\Omega)$ is quasi-homogeneous.

To show that the action of $\tilde{G}$ is proper, observe that $\Phi(G)$ preserves the level hyperplanes L_k, and acts properly on those hyperplanes, and that Γ acts properly on $\mathbb{R}_+$. Note that in general the product of two proper actions need not be proper, but since mapping $\mathbb{R}^{n+1} \to \mathbb{R}$ given by projection onto the last coordinate is $\Phi(G)$-equivariant, the product of these actions is in fact proper.

The cone $C(\Omega)$ now satisfies the hypotheses of Lemma 2, and hence there is a 1 dimensional D which is $\tilde{G}$ stable and $\mathbb{R}^{n+1} = L_0 \oplus D$. Let $\pi_1 : C(\Omega) \to L_0$ and $\pi_2 : C(\Omega) \to D$ be the canonical projections.

The images of $C(\Omega))$ under π_1 and π_2 are now sharp convex cones D^+ and Ω_0 contained in D and L_0 respectively, with both cone points being the origin of $\mathbb{R}^{n+1}$. Hence $C(\Omega) = D^+ \times \Omega_0$. Now Ω is simply a translation of Ω_0 in the direction of D^+, hence Ω is a cone. □

In [3], Goldman gave examples which showed that each of the hypotheses (properly convex and divisible) are necessary. In each of the examples below one hypothesis is missing and the domains are not cones.

1) The domain $\{(x, y) : y > x^2\}$ in $\mathbb{R}^2$ is properly convex but not a divisible domain, and is not a cone.

2) The domain $\{(x, y, z) : y > x^2\}$ in $\mathbb{R}^3$ is a divisible domain but is not properly convex, and of course is not a cone.

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