MATLAB Project # 1 – Due Date: 10/9/2008

In order to do this project you have to download the files `GE.m`, `ltrisol.m`, `utrisol.m`, `lbidisol.m`, `elim.m`, `partic.m`, `nullbasis.m` from the website: `http://www.math.umd.edu/~rhn/teaching`.

You must work each problem in a separate M-file and hand in a printout of each problem (`Project # 0` on the webpage of the course explains how to present homework). You can alternatively use the new MATLAB command `publish`. To prevent MATLAB from showing big matrices and vectors, you should add a ';' at the end of each instruction (type '`help ;`'). Please, in this problem use ';' properly and do not show the matrices and vectors in the papers you hand in. Show only the instructions, outputs and your comments and answers.

Problem 1. (25 pts) In this problem we compare different ways of computing the solution of the same linear problem. Let $n = 10$ and define the $n \times n$ *tridiagonal* matrix A and the n-vector $\mathbf{b}$ by the instructions

```
>> A = diag(2*ones(1,n)) - diag(ones(1,n-1),1) - diag(ones(1,n-1),-1)
```

(type `help ones` and `help diag` to find out how these commands work)

```
>> b = [0:1:n/2-1 n/2-1:-1:0]'   % don't forget the '
```

To solve the linear equation $A^5\mathbf{x} = \mathbf{b}$ there are at least three ways:

(a) The first immediate way boilds down to using the MATLAB command "\" and typing

```
>> x = (A^5)\b
```

(b) The second way results from observing that solving $A^5\mathbf{x} = \mathbf{b}$ is equivalent to solving $A(A(A(A(A\mathbf{x})))) = \mathbf{b}$ and thus the sequence $A\mathbf{x}_1 = \mathbf{b}$, $A\mathbf{x}_2 = \mathbf{x}_1$, $A\mathbf{x}_3 = \mathbf{x}_2$, $A\mathbf{x}_4 = \mathbf{x}_3$, $A\mathbf{x} = \mathbf{x}_4$ will give the solution. We can solve each of these five linear systems with the \ command.

(c) The third way to do this is by computing first the LU decomposition of A, using a function `tridia`, and then solving the five linear systems of (b) with `lbidisol` and `ubidisol` **without** re-decomposing the matrix A.

Solve the system $A^5x = b$ by these three ways, for which you have to write MATLAB functions `[l,u]=tridia(d,e,f)` and `x=ubidisol(u,f,b)`. Do a rough count by hand of the number of operations (flops) needed by each method as a function of n and compare. Draw conclusions.

Problem 2. (25 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ -1 & 3 & -3 \end{bmatrix}.$$

(a) Use the MATLAB function `[L,U]=GE(A)` to compute the LU decomposition of A without pivoting. Explain what happens.

(b) Write a MATLAB function `[L,U,piv]=GEpiv(A)`, by modifying `GE` as explained in class, to find the LU factorization of A with row exchanges; here `piv` is a permutation vector. Explain how to find the permutation matrix P from `piv` such that $PA = LU$. Apply to A and check that $PA = LU$.

(c) Let $\mathbf{b} = [5, 4, 3]^T$. Use `ltrisol` and `utrisol` to solve $A\mathbf{x} = \mathbf{b}$.

Problem 3. (25 pts). The *Hilbert* matrix $H_n = (h_{ij})_{i,j=1}^n$ of order n is defined by

$$h_{ij} = \frac{1}{i+j-1}.$$

This matrix is nonsingular and has an explicit inverse. However, as n increases, the condition number of H_n increases rapidly. The MATLAB functions `hilb(n)` and `invhilb(n)` give H_n and H_n^{-1} respectively. Given $\mathbf{b}_n = (1, 0, \cdots, 0)$, we want to solve $H_n \mathbf{x}_n = \mathbf{b}_n$.

(a) Solve for $n = 5, 10$ using the MATLAB command "\", and call the computed result $\mathbf{x}_n^*$.

(b) Compute the exact solution $\mathbf{x}_n = H_n^{-1}\mathbf{b}_n$, the *error* $\mathbf{e}_n = \mathbf{x}_n - \mathbf{x}_n^*$, and the *residual* $\mathbf{r}_n = \mathbf{b}_n - H_n\mathbf{x}_n^*$.

(c) Find the *condition number* $\text{cond}(H_n)$ of H_n using the command `cond`. This number gives an estimate on the expected relative accuracy of the solution: if $\text{cond}(H_n) \approx 10^t$ with $t \geq 0$, then the number of correct decimal digits in the solution is expected to be $16 - t$. How many correct decimal digits do you expect for $n = 5, 10$?

Problem 4. (25 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 12 & 8 & 9 & 10 \\ 2 & 4 & 4 & 5 & 6 \end{bmatrix}.$$

(a) Use `rref` to find the reduced row echelon form R and the pivot columns for A.

(b) Use `elim` to find the reduced row echelon form R, and the elimination matrix E that puts A into reduced row echelon form R: $R = EA$.

(c) Use the results of (a) and (b) to find a basis for the solution space of $A\mathbf{x} = 0$.

(d) Use `nulbasis` to find a basis for $N(A)$. How does this result relate to that found in (c)?

(e) What is the general solution to $A\mathbf{x} = 0$?

(f) Use `rank` to find the rank of A. Relate to the dimensions of A and $N(A)$.

(g) Find the condition on $\mathbf{b} = [b_1, b_2, b_3]^T$ that ensures $A\mathbf{x} = \mathbf{b}$ has solutions. To do this perform row reduction on $[A \quad \mathbf{b}]$ by hand calculation.

(h) Use `partic` to find a particular solution to $A\mathbf{x} = [0, 5, 1]^T$. Does $[0, 5, 1]^T$ satisfy the condition you found in part (g)?

(i) Use the result in (e) and (h) to write the general solution to $A\mathbf{x} = [0, 5, 1]^T$.