1 (15 pts). Lax-Wendroff Scheme: (a) Problem 12.2 of Larsson and Thomée.
(b) Show that the scheme in p.191 of Larsson and Thomée can be written as
\[ \frac{U_{j}^{n+1} - U_{j}^{n}}{k} - \frac{U_{j+1}^{n} - U_{j-1}^{n}}{2h} - \frac{ka^{2}}{2} \frac{U_{j+1}^{n} - 2U_{j}^{n} + U_{j-1}^{n}}{h^{2}} = 0. \]
(c) Prove that the method is of second order by direct use of Taylor expansion.

2 (20 pts). MATLAB. (a) Consider the linear 1st order hyperbolic PDE \( u_t + u_x = 0 \) in the space interval \( \Omega = (-1, 1) \) and time interval \( (0, 0.5) \). Let the initial conditions be either \( v(x) = \max(1 - 4|x + 0.25|, 0) \), \( v(x) = \begin{cases} 1 & x \leq -0.25 \\ 0 & x > -0.25 \end{cases} \) and boundary conditions \( U_0 = v(-1) \) at \( x_0 = -1 \) (inflow) and \( U_J^j = U_{J-1}^j \) at \( x_J = 1 \) (outflow). Implement the following schemes for arbitrary \( k, h \): (i) Upwind, (ii) Lax-Friedrichs, (iii) Lax-Wendroff.
(b) Run the programs with \( k = 0.8 \) and \( h = 0.01, 0.005, 0.0025 \), and plot both the computed and true solutions at \( t = 0.5 \).
(c) Compare the methods in terms of the smearing effect around corners and jumps and the presence of oscillations.

3 (15 pts) Convex flux. Consider the Cauchy problem for Burgers’ equation \( u_t + uu_x = 0 \) with initial condition \( u_0(x) = 1 \) for \( |x| > 1; \) \( u_0(x) = |x| \) for \( |x| < 1 \).
(a) Sketch the characteristics in the \((x,t)\) plane. Find a classical solution (continuous and piecewise \(C^1\)). Determine the time of breakdown (shock formation).
(b) Find a weak solution globally for \( t > 0 \), containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then, use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by \( (x = s(t), t) \) in the \((x,t)\)-plane.

4 (15 pts). Nonconvex flux. The Buckley-Leverett equation is a simple model for two-phase fluid flow in a porous medium with flux
\[ f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}. \]
In secondary oil recovery, water is pumped into some wells to displace the oil remaining in the underground rocks. Therefore \( u \) represents the saturation of water, namely the percentage of water in the water-oil fluid, and varies between 0 and 1. Find the entropy solution to the Riemann problem with initial states
\( u_0(x) = 1 \) for \( x < 0; \) \( u_0(x) = 0 \) for \( x > 0 \).
Hint: The line through the origin that is tangent to the graph of \( f \) on the interval \([0,1]\) has slope \( 1/(\sqrt{3} - 1) \) and touches the curve at \( u = 1/\sqrt{3} \).

5 (10 pts). Engquist-Osher Scheme. Let \( f(0) = 0 \) and consider the numerical flux
\[ F(u_L, u_R) = \int_0^{u_L} \max(f'(s), 0)ds + \int_0^{u_R} \min(f'(s), 0)ds \]
(a) Show that the resulting method is consistent and monotone.
(b) Show that $F(u_L, u_R)$ can be equivalently written as

$$F(u_L, u_R) = \frac{1}{2} \left( f(u_L) + f(u_R) - \int_{u_L}^{u_R} |f'(s)| ds \right).$$

(c) If $f(u) = u^2/2$ show that $F(u_L, u_R)$ can be equivalently written as

$$F(u_L, u_R) = \frac{1}{2} \left( \max(u_L, 0)^2 + \min(u_R, 0)^2 \right).$$

6 (10 pts). Lax-Friedrichs Scheme. This method reads

$$U^k_{j+1} - \frac{U^k_{j+1} + U^k_{j-1}}{2} \frac{\Delta t}{h} + \frac{f(U^k_{j+1}) - f(U^k_{j-1})}{2h} = 0.$$ 

(b) Show that this method is consistent and monotone.

7 (15 pts). Godunov Scheme. This method comes from solving exactly 1d Riemann problems with piecewise constant data (see Lucier's notes).

(a) Assume that the flux $f$ is convex. Show that the corresponding numerical flux can be written as

$$F(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u) & u_L \leq u_R \\ \max_{u_R \leq u \leq u_L} f(u) & u_R < u_L. \end{cases}$$

This formula is still valid for any Lipschitz flux regardless of convexity.

(b) Show that this method is consistent and monotone.