AMSC 612 Fall 2008

NUMERICAL METHODS FOR PDE

HOMEWORK # 4

(Pbs 1-2 due Tu Nov 13, Pbs 2-6 due Th Nov 20)

1 (15 pts). Use the MATLAB code `fem` from the website `morin.pedro.googlepages.com` to solve the following problem. Let $\Omega = [-1,1]^2 \backslash [0,1] \times [0,-1]$ be an L-shaped domain. Let u, given in polar coordinates (r, θ),

$$u(x,y) = r^{2/3} \sin(2\theta/3)$$

be the exact solution of the Dirichlet boundary value problem $-\Delta u = f$ in Ω.
(a) Generate the data files `vertex_coordinates.txt`, `elem_vertices.txt`, `dirichlet.txt`, and `neumann.txt` using `gen_mesh_L_shape.m` for uniform refinement with meshsize $h = \frac{1}{N} = 2^{-k}$ and $k = 2, 3, 4, 5, 6, 7$. Find the corresponding solutions u_h.
(b) Show that the stiffness matrices for these meshes and those for finite differences with a 5-point stencil coincide.
(c) Find the errors $|u - u_h|_{H_0^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$, and plot them in a log-log plot. Determine the experimental order of convergence and comprae with theory. Draw conclusions.

2 (20 pts). Consider the following second order initial value problem modeling a spring-dashpot system:

$$y'' + 101y + 100y = \sin x, \qquad y(0) = 2, y'(0) = 0. \tag{1}$$

(a) Solve this problem by hand. To this end find first the solution to the homogeneous equation (natural modes), and next a particular solution using the method of undetermined coefficients. Explain whether the problem is *stiff* or not.
(b) Convert (1) into a first order system, and write the forward Euler (FE), backward Euler (BE) and Trapezoid methods (TM).
(c) Write MATLAB programs that implement FE, BE, and TM with step-sizes $h = 10^{-k}$ for $k = 1, 2, 3$ on the interval $(0, 10)$.
(d) Find the error between computed solutions of (c) and exact solution of (a) at $t_n = nh$ and plot the results. Explain the results in terms of ℓ^∞-stability, and draw conclusions.

3 (15 pts). Problem 8.5 of Larsson and Thomée.

4 (15 pts). Problem 8.7 of Larsson and Thomée.

5 (15 pts). Problem 9.4 of Larsson and Thomée. To derive the 5-point stencil proceed as follows. Suppose that $u \in C^6$ and show an error estimate for $u''(0) - D_h^2(0)$ where

$$D_h^2(0) = h^{-2}\big(u(-h) - 2u(0) + u(h)\big),$$

is the usual 3-point stencil; keep all terms in the error expression up to order 4. Combine now $D_h^2(0)$ with $D_{2h}^2(0)$ in order to eliminate the leading error term of order 2. This process is called *extrapolation.*

6 (20 pts). Problem 9.11 of Larsson and Thomée in $\Omega = (-1, 1)$. To this end
(a) Combine Problems 4 of HW#1 and 2 of HW#4 to implement the forward Euler (FE), backward Euler (BE) and Crank-Nicolson methods (CN) for (9.19) with initial value $v(x)$ and boundary value $g(x, t)$ given by

$$v(x) = \sin(\pi x) - \sin(3\pi x), \quad g(\pm 1, t) = 0,$$
$$v(x) = \mathrm{sign(x)}, \quad g(x, t) = \mathrm{erf}(\pm 1/\sqrt{4t});$$

erf is the solution of the heat equation in the whole line with initial condition $\mathrm{sign}(x)$.
(b) Let $h = 1/10$. Compute the discrete solution with $k = 1/100, 1/300, 1/600$ for FE and BE and with $k = 1/10$ for CN.
(c) Find the exact solutions and compute the errors at $(0.5, 1)$. Draw conclusions.