AMSC 612 Fall 2008

NUMERICAL METHODS FOR PDE

HOMEWORK # 5

due Fr 12/19

1 (25 pts). *FEM for the Heat Equation*: (a) Modify the MATLAB code `fem` of HW#4-Pb1 by adding a loop $1 \leq n \leq N$ to account for a backward Euler discretization of the time variable.
(b) Let $\Omega = (0,1)^2$ and $T = 1$. Let a Neumann condition g_N be imposed on the side $x = 1$, and a Dirichlet condition g_D on the rest of the boundary $\partial\Omega$. Let

$$u(x, y, t) = \sin(3\pi x)e^{-y-2t}.$$

be the exact solution. Find g_D and the forcing $f = \partial_t - \Delta u$.
(c) Compute the discrete solution U at $T = 1$, along with the L^2, H^1, and L^∞ errors. To this end, use the relation $k = h^2$ between time-step and meshsize, and find the discrete solution for meshsizes $h = 2^{-j}$ with $j = 3, 4, 5, 6$. Plot the discrete solution at $T = 1$ for $h = 2^{-4}$ and plot the three errors in terms of h in a log-log scale. Verify the decay $\|u(T) - U(T)\| \approx h^s$ and relate s to theory.

2 (25 pts). *Lax-Wendroff Scheme*: (a) Problem 12.2 of Larsson and Thomée.
(b) Show that the scheme in p.191 of Larsson and Thomée can be written as

$$\frac{U_j^{n+1} - U_j^n}{k} - a\frac{U_{j+1}^n - U_{j-1}^n}{2h} - \frac{ka^2}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = 0.$$

(c) Prove that the method is of second order by direct use of Taylor expansion.

3 (25 pts). (a) Consider the linear 1st order hyperbolic PDE $u_t + u_x = 0.$ in the space interval $\Omega = (-1, 1)$ and time interval $(0, 0.5)$. Let the initial conditions be either

$$v(x) = \max(1 - 4|x + 0.25|, 0), \qquad v(x) = \begin{cases} 1 & x \leq -0.25 \\ 0 & x > -0.25 \end{cases}$$

and boundary conditions $U_0 = v(-1)$ at $x_0 = -1$ (inflow) and $U_J^n = U_{J-1}^n$ at $x_J = 1$ (outflow). Implement the following schemes for arbitrary k, h: (i) Upwind, (ii) Lax-Friedrichs, (iii) Lax-Wendroff.
(b) Run the programs with $\frac{k}{h} = 0.8$ and $h = 0.01, 0.005, 0.0025$, and plot both the computed and true solutions at $t = 0.5$.
(c) Compare the methods in terms of the smearing effect around corners and jumps and the presence of oscillations.

4 (25 pts). Problem 13.3 of Larsson and Thomée. To compare with the method (13.5) in Larsson and Thomée, consider the time-discrete scheme only (i.e. without space discretization).