1 Introduction

In a recent paper, Heinig and Maligranda [2] have applied Chebyshev's inequality for decreasing(increasing) functions f, g

$$\int_0^x pf \int_0^x pg \leq \int_0^x pfg \int_0^x p$$

in a variety of situations. Heinig asked if it was possible to extend the result to the class of functions whose averages decrease. I give a new sufficient condition which includes this class of functions.

The inequality is naturally expressed in terms of an averaging operator A which is defined for suitable f and a continuous $p > 0$ by

$$Af(x) = \frac{\int_0^x pf}{\int_0^x p}.$$

Steffensen [3] has given a necessary condition for Chebyshev's inequality. If one of the functions, say f, is increasing, Steffensen has shown that Chebyshev's inequality on $[0, a]$ implies $Ag(t) \leq Ag(a)$ for each $t \in [0, a]$. In particular, if we assume that Chebyshev's inequality holds for each a in $[0, \infty)$, and f is increasing, it follows from Steffensen's result that Ag must be increasing. This shows that our sufficient condition is also necessary for the inequality to hold globally when one of the functions is monotone. If it were possible to extend Steffensen's result to the case Af increasing, it would follow that our conditions are near the correct ones, at least for the global inequality. A key step in Steffensen's proof is false for functions whose average is increasing; we do not know if the result can be generalized.

2 Main Results

Throughout, we will use the notation

$$Af(x) = \frac{1}{\int_0^x p} \int_0^x p(t)f(t)\,dt.$$

We will assume that $p > 0$ and that p is continuous. By Chebyshev's inequality, we understand the pointwise inequality

$$Af(x)Ag(x) \leq A(fg)(x)$$

for all $x > 0$, which can also be written

$$\int_0^x pf \int_0^x pg \leq \int_0^x p \int_0^x pfg.$$

Note that if we consider the functions $F(x) = f(a+(b-a)x), G(x) = g(a+(b-a)x), P(x) = p(a+(b-a)x)$ for $0 \leq x \leq 1$, apply the results and make the obvious change of variables, we will also get results for the inequality

$$\int_a^b pf \int_a^b pg \leq \int_a^b p \int_a^b pfg.$$

Lemma 1 *$Af \uparrow \Leftrightarrow f \geq Af$; $Af \downarrow \Leftrightarrow f \leq Af$.*

Proof. We compute that

$$(Af)'(x) = \frac{p(x)(f(x) - Af(x))}{\int_0^x p}$$

from which the result follows. ∎

Lemma 2 *If $f \uparrow$, then $Af \uparrow$. If $f \downarrow$, then $Af \downarrow$.*

Lemma 3 *If $f(x) \leq g(x)$ for $0 \leq x \leq a$, then $Af(x) \leq Ag(x)$ for $0 \leq x \leq a$.*

Theorem 1 *If*

$$(f - Af)(g - Ag) \geq 0 \tag{1}$$

for $0 \leq x \leq a$, Chebyshev's inequality holds for $0 \leq x \leq a$.

Proof. Let $H(x) = A(fg)(x) - Af(x)Ag(x)$. One sees that $H'(x) = A(fg)' - Af'Ag - Ag'Af$ and since

$$(Af)'(x) = \frac{p(f - Af)}{\int_0^x p}$$

we have

$$H'(x) = \frac{p(x)}{\int_0^x p}\left(fg - A(fg) - (f - Af)Ag - Af(g - Ag)\right)$$

$$= \frac{p(x)}{\int_0^x p}\left(f(g - Ag) - A(fg) + AfAg - Af(g - Ag)\right)$$

$$= \frac{p(x)}{\int_0^x p}\left((f - Af)(g - Ag) - H\right).$$

Now if $(f-Af)(g-Ag)\geq 0$ for $0\leq x\leq a$, then

$$(\int_0^x p)H' + pH = ((\int_0^x p)H)' \geq 0,$$

which implies that $(\int_0^x p)H \uparrow$ for $0\leq x\leq a$. Since $H(0)=0$, $H(x)\geq 0$ for all $0\leq x\leq a$.

Remark. Note that we have shown $((\int_0^x p)H)' = (f-Af)(g-Ag)$ and hence, if $(f-Af)(g-Ag)\leq 0$, then $H(x)\leq 0$ for $x\geq 0$. It also follows that $(\int_0^x p)H$ is increasing iff $(f-Af)(g-Ag)\geq 0$.

Now if $Af\uparrow$ and $Ag\uparrow$, then $f\geq Af$ and $g\geq Ag$, which means that $(f-Af)(g-Ag)\geq 0$ and we get the same inequality if $Af\downarrow$ and $Ag\downarrow$ since then $f\leq Af$ and $g\leq Ag$ and the product is still nonnegative.

We can also give a purely algebraic proof of this inequality. It requires us to note that A is an invertible operator with inverse $A^{-1}f(x) = ((\int_0^x p)f)'/p(x)$ and to note the following inequality which follows from integration by parts.

Lemma 4

$$A[fAg+gAf] = AfAg + A[AfAg].$$

This follows from the obvious integration by parts or by applying A^{-1} to both sides and computing the right hand side.

Now suppose that $0\leq (f-Af)(g-Ag)$. The right hand side of this expression is

$$fg-(fAg+gAf)+AfAg = fg - A^{-1}(AfAg) - AfAg + AfAg$$

and we take the middle term with A^{-1} to the left hand side of the equation which shows

$$A^{-1}(AfAg)\leq fg.$$

The result follows by applying Lemma 3.

Lemma 5 *If $Af\downarrow, Ag\downarrow$, then $A(fg)\downarrow$.*

Proof. It follows from the above theorem that $AfAg\leq A(fg)$. However, by Lemma 1, since $Af\downarrow$, $f\leq Af$ and hence, $fg\leq AfAg\leq A(fg)$, and thus by Lemma 1 again, $A(fg)\downarrow$. ■

Thus the set of nonnegative functions whose averages decrease is a subalgebra of the set of all nonnegative functions.

Theorem 2 *If $Af \downarrow, Ag \downarrow$, then $A^m f A^n g \leq A^{\max(m,n)} fg$.*
If $Af \uparrow, Ag \uparrow$, then $A^m f A^n g \leq A^{\min(m,n)}(fg)$.

Proof. Observe first that $A^m f \downarrow$ or $A^m f \uparrow$ follows by induction. Let us consider the case of $Af \downarrow$ which is the case $m = 1$ of the induction. If $A^{m_0} f \downarrow$, then by Lemma 1

$$A(A^{m_0} f) \leq A^{m_0} f$$

and hence, by Lemma 3

$$A(A^{m_0+1} f) \leq A(A^{m_0} f)$$

which proves, by Lemma 1 again, that $A^{m_0+1} f \downarrow$.

Consider first the case in which $Af \downarrow, Ag \downarrow$. By our hypothesis, $Af Ag \leq A(fg)$, and thus by induction, $A^m f A^m g \leq A^m(fg)$. One sees this because if the inequality is true for m_0, then

$$A^{m_0+1} f A^{m_0+1} g = A(A^{m_0} f) A(A^{m_0} g) \leq A(A^{m_0} f A^{m_0} g)$$

because $A^{m_0} f, A^{m_0} g \downarrow$, and this is

$$\leq A(A^{m_0}(fg))$$

by the induction hypothesis. From the definition of composition, this latter inequality is the desired inequality.

Now if $m \geq n$, we write

$$A^m f A^n g = A^n (A^{m-n} f) A^n g \leq A^n((A^{m-n} f) g)$$

by the above result. However, since $Ag \downarrow$, Lemma 1 shows $g \leq Ag$ and by Lemma 3 we obtain $g \leq Ag \leq A^2 g \ldots$, which allows us to estimate the last term by

$$A^n(A^{m-n} f A^{m-n} g) \leq A^n(A^{m-n}(fg)) = A^m(fg).$$

Either using symmetry or noting that if $m \leq n$, we proceed as above but using $A^m f A^n g = A^m f A^m (A^{n-m} g)$ gives the result in the second case.

If $Af \uparrow$, we use the fact that $Af \leq f$, and again by induction and the fact that A is increasing, $\ldots \leq A^2 f \leq Af \leq f$, to write for $m \geq n$,

$$A^m f A^n g = A^n(A^{m-n} f) A^n g \leq A^n((A^{m-n} f) g) \leq A^n(fg).$$

Since a similar argument works for $m \leq n$, we are done. ■

Theorem 3

$$A^m f(x) = \frac{\frac{1}{(m-1)!}\int_0^x p(t) f(t) \ln^{m-1}\left(\frac{\int_0^x p}{\int_0^t p}\right) dt}{\int_0^x p}.$$

PROOF. The formula is correct for $m = 1$ and can be checked also for $m = 2$. Assume that it is valid for $m = m_0$. We compute

$$A^{m_0+1} f(x) = A^{m_0} A f(x)$$

$$= \frac{\frac{1}{(m-1)!}\int_0^x p(t) A f(t) \ln^{m-1}\left(\frac{\int_0^x p}{\int_0^t p}\right) dt}{\int_0^x p}$$

If we multiply the expression through by $\int_0^x p$, we see that the numerator of the fraction is

$$\frac{1}{(m-1)!}\int_0^x \frac{p(t)}{\int_0^t p(u)\, du}\left(\int_0^t p(s) f(s)\, ds\right) \ln^{m-1}\left(\frac{\int_0^x p}{\int_0^t p}\right) dt$$

and if we interchange the order of integration, we obtain

$$\frac{1}{(m-1)!}\int_0^x p(s) f(s) \int_s^x \frac{p(t)}{\int_0^t p(u)\, du} \ln^{m-1}\left(\frac{\int_0^x p}{\int_0^t p}\right) dt.$$

We change variables in this expression by letting

$$v = \frac{\int_0^x p}{\int_0^t p}, \frac{dv}{v} = -\frac{p(t)}{\int_0^t p}\, dt$$

and we obtain for the numerator of the expression with which we began

$$\frac{1}{(m-1)!}\int_0^x p(s) f(s) \int_{\frac{\int_0^x p}{\int_0^s p}}^1 \ln^{m-1} v\, (-\frac{dv}{v})$$

$$= \frac{1}{m!}\int_0^x p(s) f(s) \ln^m\left(\frac{\int_0^x p}{\int_0^s p}\right) ds.$$

This is the result we were trying to prove. ■

Theorem 4 *If $Af\downarrow, Ag\downarrow$, then $\forall\tau\in R^+$,*

$$\left(\frac{1}{\int_0^x p}\int_0^x p(t)f(t)(\frac{\int_0^x p}{\int_0^t p})^{\frac{1}{\tau}}\,dt\right)\left(\frac{1}{\int_0^x p}\int_0^x p(s)g(s)(\frac{\int_0^x p}{\int_0^s p})^{\frac{1}{\tau}}\,ds\right)$$

$$\leq\frac{1}{\int_0^x p}\int_0^x(pfg)(u)(\frac{\int_0^x p}{\int_0^u p})^{\frac{1}{\tau}}(1+\frac{1}{\tau}\ln\frac{\int_0^x p}{\int_0^u p})\,du.$$

This is one of several theorems we shall give which are proved using the same process. While the inequality is not as sharp as possible, it is given because it holds for all $\tau>0$.

Proof. By Theorem 2, we have since $Af\downarrow$, $Ag\downarrow$

$$A^m fA^n g\leq A^{\max(m,n)}(fg)\leq A^{m+n-1}(fg)$$

because $m,n\geq 1$ and Lemma 5 implies that

$$fg\leq A(fg)\leq A^2(fg)\leq A^3(fg)\ldots.$$

Multiply this by

$$\frac{1}{\tau^{m+n-2}}$$

and sum over all $m,n=1$ to ∞. We obtain

$$\sum_{m=1,n=1}^{\infty}\frac{1}{\tau^{m+n-2}}A^m fA^n g\leq\sum_{m=1,n=1}^{\infty}\frac{1}{\tau^{m+n-2}}A^{m+n-1}(fg).$$

The left hand side is

$$\sum_{m=1}^{\infty}\frac{1}{\tau^{m-1}}A^m f\sum_{n=1}^{\infty}\frac{1}{\tau^{n-1}}A^n g.$$

These terms are handled similarly. We have

$$\sum_{m=1}^{\infty}\frac{1}{\tau^{m-1}}A^m f=\frac{1}{\int_0^x p}\sum_{m=1}^{\infty}\int_0^x p(t)f(t)\frac{\ln^{m-1}\left(\frac{\int_0^x p}{\int_0^t p}\right)}{\tau^{m-1}(m-1)!}\,dt$$

$$=\frac{1}{\int_0^x p}\int_0^x p(t)f(t)\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\left(\ln(\frac{\int_0^x p}{\int_0^t p})^{1/\tau}\right)^{m-1}\,dt$$

$$= \frac{1}{\int_0^x p} \int_0^x p(t) f(t) e^{\ln(\frac{\int_0^x p}{\int_0^t p})^{1/\tau}} \, dt$$

$$= \frac{1}{\int_0^x p} \int_0^x p(t) f(t) \left(\frac{\int_0^x p}{\int_0^t p} \right)^{1/\tau} dt.$$

The right hand side is

$$\sum_{m,n=1}^{\infty} \frac{1}{\tau^{m+n-2}} A^{m+n-1}(fg)$$

$$= \sum_{k=1}^{\infty} \frac{1}{\tau^{k-1}} A^k(fg) \sum_{m+n-1=k} 1 = \sum_{k=1}^{\infty} \frac{k}{\tau^{k-1}} A^k(fg)$$

$$= \frac{1}{\int_0^x p} \int_0^x (pfg)(u) \sum_{k=1}^{\infty} \frac{k \ln^{k-1}(\frac{\int_0^x p}{\int_0^u p})}{(k-1)! \tau^{k-1}} \, du.$$

We observe that

$$\sum_{k=1}^{\infty} \frac{k}{(k-1)!} z^{k-1} = \sum_{k=1}^{\infty} \frac{(k-1)+1}{(k-1)!} z^{k-1}$$

$$= \sum_{k=2}^{\infty} \frac{1}{(k-2)!} z^{k-1} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} z^{k-1} = z e^z + e^z,$$

and thus the right hand side is

$$\frac{1}{\int_0^x p} \int_0^x (pfg)(u) (\frac{\int_0^x p}{\int_0^u p})^{1/\tau} (1 + \frac{1}{\tau} \ln \frac{\int_0^x p}{\int_0^u p}) \, du. \blacksquare$$

Next we consider the case $Af \downarrow$, $Ag \downarrow$ where we attempt to use the full strength of the inequality

$$A^m f A^n g \leq A^{\max(m,n)}(fg).$$

The result is considerably more complicated in this case.

Theorem 5 *If $Af \downarrow$, $Ag \downarrow$, then the expression*

$$\left(\frac{1}{\int_0^x p} \int_0^x p(t) f(t) (\frac{\int_0^x p}{\int_0^t p})^{1/\tau} \, dt \right) \left(\frac{1}{\int_0^x p} \int_0^x p(s) g(s) (\frac{\int_0^x p}{\int_0^s p})^{1/\lambda} \, ds \right)$$

is bounded by the following expressions: for $\lambda, \tau > 1$,

$$\frac{1}{\int_0^x p} \int_0^x (pfg)(u) \left\{ (\frac{\int_0^x p}{\int_0^u p})^{1/\lambda\tau} + \frac{\tau}{\tau - 1} (\frac{\int_0^x p}{\int_0^u p})^{1/\lambda} + \frac{\lambda}{\lambda - 1} (\frac{\int_0^x p}{\int_0^u p})^{1/\tau} \right\} dx. \tag{2}$$

For $\lambda < 1, \tau < 1$, it is bounded by

$$(1 + \frac{\tau}{\tau - 1} + \frac{\lambda}{\lambda - 1}) \frac{1}{\int_0^x p} \int_0^x (pfg)(u) (\frac{\int_0^x p}{\int_0^u p})^{\frac{1}{\lambda\tau}} \, du. \tag{3}$$

For $\lambda < 1, \tau > 1$, it is bounded by

$$\frac{1}{\int_0^x p} \int_0^x pfg(u) \left\{ (1 + \frac{\lambda}{\lambda - 1}) (\frac{\int_0^x p}{\int_0^u p})^{\frac{1}{\tau\lambda}} + \frac{\tau}{\tau - 1} (\frac{\int_0^x p}{\int_0^u p})^{\frac{1}{\lambda}} \right\} du. \tag{4}$$

For $\lambda = 1, \tau = 1$, it is bounded by

$$\frac{1}{\int_0^x p} \int_0^x pfg(u) (\frac{\int_0^x p}{\int_0^u p}) (1 + 2 \ln \frac{\int_0^x p}{\int_0^u p}) \, du. \tag{5}$$

Proof. We multiply the inequality $AfAg \leq A^{\max(m,n)}(fg)$ by

$$\frac{1}{\tau^{m-1} \lambda^{n-1}}$$

and sum. Instead of getting the term which we wrote

$$\sum_{m=1}^{\infty} \frac{1}{\tau^{m-1}} A^m f \sum_{n=1}^{\infty} \frac{1}{\tau^{n-1}} A^n g,$$

the term is

$$\sum_{m=1}^{\infty} \frac{1}{\tau^{m-1}} A^m f \sum_{n=1}^{\infty} \frac{1}{\lambda^{n-1}} A^n g$$

and the individual terms are handled as above giving the left hand side of the expression.

The right hand side of the expression is

$$\sum_{m,n=1}^{\infty} \frac{1}{\tau^{m-1} \lambda^{n-1}} A^{\max(m,n)}(fg)$$

$$= \sum_{k=1}^{\infty} A^k(fg) \sum_{\max(m,n)=k} \frac{1}{\tau^{m-1} \lambda^{n-1}}.$$

The set $\{(m,n)|\max(m,n)=k\}$ is a finite set. It consists of

$$\{(k,k)\cup\{(j,k)|j<k\}\cup\{(k,j)|j<k\}\}.$$

For $k=1$ there is only the term $A(fg)$. For $k>1$, the inner sum is

$$\frac{1}{\tau^{k-1}\lambda^{k-1}}+\frac{1}{\lambda^{k-1}}\left(\sum_{j=1}^{k-1}\frac{1}{\tau^{j-1}}\right)+\frac{1}{\tau^{k-1}}\left(\sum_{j=1}^{k-1}\frac{1}{\lambda^{j-1}}\right)$$

$$=\frac{1}{\tau^{k-1}\lambda^{k-1}}+\frac{1}{\lambda^{k-1}}\left\{1+\frac{1}{\tau}+\ldots+\frac{1}{\tau^{k-1}}\right\}+\frac{1}{\tau^{k-1}}\left\{1+\ldots+\frac{1}{\lambda^{k-1}}\right\}.$$

Each series is a finite geometric series and gives rise to a term of the form (A_k-B_k). In each of the cases, the result follows by estimating the term from above by A_k and dropping the second term.

The case $\lambda=1,\tau=1$ deserves special mention. The above described set has $2k-1$ elements and thus the term on the right hand side becomes

$$\sum_{k=1}^{\infty}(2k-1)A^k(fg)=\sum_{k=1}^{\infty}\frac{1}{\int_0^x p}\int_0^x pfg(u)\frac{\ln^{k-1}(\frac{\int_0^x p}{\int_0^u p})}{(k-1)!}(2k-1)\,du$$

$$=\frac{1}{\int_0^x p}\int_0^x pfg(u)\sum_{k=1}^{\infty}\frac{2(k-1)+1}{(k-1)!}\ln^{k-1}(\frac{\int_0^x p}{\int_0^u p})\,du$$

$$=\frac{1}{\int_0^x p}\int_0^x pfg(u)\left(\sum_{k=2}^{\infty}\frac{2}{(k-2)!}\ln^{k-1}(\frac{\int_0^x p}{\int_0^u p})+\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\ln^{k-1}(\frac{\int_0^x p}{\int_0^u p})\right)du$$

$$=\frac{1}{\int_0^x p}\int_0^x pfg(u)(\frac{\int_0^x p}{\int_0^u p})(2\ln\frac{\int_0^x p}{\int_0^u p}+1)\,du.$$

This is the desired result in this case. ■

Next let us consider the case $Af\uparrow, Ag\uparrow$. A similar argument to the above allows us to prove the next theorem.

Theorem 6 *If $Af\uparrow, Ag\uparrow$, then for any $\lambda>1,\tau>1$, we have*

$$\left(\frac{1}{\int_0^x p}\int_0^x p(t)f(t)(\frac{\int_0^x p}{\int_0^t p})^{1/\tau}\,dt\right)\left(\frac{1}{\int_0^x p}\int_0^x p(s)g(s)(\frac{\int_0^x p}{\int_0^s p})^{1/\lambda}\,ds\right)$$

$$\le\left\{1+\frac{1}{\lambda-1}+\frac{1}{\tau-1}\right\}\frac{1}{\int_0^x p}\int_0^x(pfg)(u)(\frac{\int_0^x p}{\int_0^u p})^{1/\lambda\tau}\,du.$$

Proof. Theorem 2 shows that under the hypotheses we have

$$A^m f A^n g \leq A^{\min(m,n)}(fg)$$

which we multiply by

$$\frac{1}{\tau^{m-1}\lambda^{n-1}}$$

and sum over all m, n. The same argument as above gives the left hand side of

$$\left(\frac{1}{\int_0^x p}\int_0^x p(t)f(t)(\frac{\int_0^x p}{\int_0^t p})^{1/\tau}\,dt\right)\left(\frac{1}{\int_0^x p}\int_0^x p(s)g(s)(\frac{\int_0^x p}{\int_0^s p})^{1/\lambda}\,ds\right).$$

The right hand side is

$$\sum_{m,n=1}^{\infty}\frac{1}{\tau^{m-1}\lambda^{n-1}}A^{\min(m,n)}(fg) = \sum_{k=1}^{\infty}A^k(fg)\sum_{\min(m,n)=k}\frac{1}{\tau^{m-1}\lambda^{n-1}}.$$

It is easy to see that

$$\{(m,n)|\min(m,n)=k\} = \{(k,k)\}\cup\{(k,j)|j>k\}\cup\{(j,k)|j>k\}.$$

We can write the inner sum on the right hand side as

$$\sum_{\min(m,n)=k}\frac{1}{\tau^{m-1}\lambda^{n-1}} = \frac{1}{\tau^{k-1}\lambda^{k-1}} + \sum_{j>k}\frac{1}{\tau^{k-1}\lambda^{j-1}} + \sum_{j>k}\frac{1}{\tau^{j-1}\lambda^{k-1}}$$

$$= \frac{1}{\tau^{k-1}\lambda^{k-1}} + \frac{1}{\tau^{k-1}}\left\{\frac{1}{\lambda^k} + \frac{1}{\lambda^{k+1}} + \dots\right\} + \frac{1}{\lambda^{k-1}}\left\{\frac{1}{\tau^k} + \frac{1}{\tau^{k+1}} + \dots\right\}$$

$$= \frac{1}{\tau^{k-1}\lambda^{k-1}} + \frac{1}{\lambda^k\tau^{k-1}}\left\{1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \dots\right\} + \frac{1}{\lambda^{k-1}\tau^k}\left(1 + \frac{1}{\tau} + \dots\right).$$

The series converge only if $\lambda > 1$ and $\tau > 1$. When those conditions hold, we obtain

$$\sum_{\min(m,n)=k}\frac{1}{\tau^{m-1}\lambda^{n-1}}$$

$$= \frac{1}{\tau^{k-1}\lambda^{k-1}}\left\{1 + \frac{1}{\lambda}(\frac{1}{1-\frac{1}{\lambda}}) + \frac{1}{\tau}(\frac{1}{1-\frac{1}{\tau}})\right\}$$

which means that the right hand side is

$$\left\{1 + \frac{1}{\lambda - 1} + \frac{1}{\tau - 1}\right\}\sum_{k=1}^{\infty}\frac{1}{\tau^{k-1}\lambda^{k-1}}A^k(fg)$$

and by a similar argument to the above utilized arguments, we obtain the theorem. ■

Remark. We can let τ or λ go to ∞ in the above inequalities to obtain

$$\left(\frac{1}{\int_0^x p}\int_0^x p(t)f(t)(\frac{\int_0^x p}{\int_0^t p})^{1/\tau}\,dt\right)\left(\frac{1}{\int_0^x p}\int_0^x p(s)g(s)\,ds\right)$$
$$\leq \left\{1+\frac{1}{\tau-1}\right\}\frac{1}{\int_0^x p}\int_0^x (pfg)(u)\,du, \forall\tau>1.$$

The limit of this expression as $\tau\to\infty$ is the Chebyshev inequality. This is also true for the previous inequalities.

3 Relation with known results

We next discuss the relation between the condition (1) and the known sufficient and necessary conditions for Chebyshev's inequality. One of those conditions is the condition that f, g be similarly ordered.

Definition 1 *f, g are similarly ordered if for all $s, t \geq 0$, we have*

$$(f(s)-f(t))(g(s)-g(t)) \geq 0. \tag{6}$$

The proof of Theorem 1 shows that $(f-Af)(g-Ag)\geq 0$ iff $(\int_0^x p)H\uparrow$. If f, g are similarly ordered, we can obtain a weaker conclusion.

Theorem 7 *If f, g are similarly ordered, then $(\int_0^x p)^2 H\uparrow$.*

PROOF. Write the fact that f, g are similarly ordered.

$$(f(s)-f(t))(g(s)-g(t)) \geq 0$$

multiply by $p(s)$ and integrate with respect to s from 0 to t. We obtain

$$\int_0^t pfg(s) - f(t)\int_0^t p(s)g(s)\,ds - g(t)\int_0^t p(s)f(s)\,ds + fg(t)\int_0^t p \geq 0.$$

On dividing by $\int_0^t p$ this can be written

$$A(fg)(t) - f(t)Ag(t) - g(t)Af(t) + fg(t) \geq 0$$

and collecting terms, we have

$$(f - Af)(g - Ag) + H \geq 0$$

where $H = A(fg) - AfAg$. However, since $(f - Af)(g - Ag) = (\int_0^x p)H' + pH$, we see that we have

$$(\int_0^x p)H' + 2pH \geq 0$$

and hence, $(\int_0^x p)^2 H \uparrow$ which proves the result.

Note that according to [2], Chebyshev proved his result for functions f, g that satisfy $sgn\frac{df}{dx} = sgn\frac{dg}{dx}$ on $[a, b]$. If f, g are similarly ordered, this condition holds as is seen by dividing (6) by $(s-t)^2$ and letting $s \to t$. It is easy to give examples of functions that satisfy Chebyshev's condition but are not similarly ordered.

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