Properties of BMO functions whose reciprocals are also BMO

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The main result says that a non-negative BMO-function w, whose reciprocal is also in BMO, belongs to $\bigcap_{p>1} A_p$,and that an arbitrary $u \in BMO$ can be written as $u = w - 1/w$, for w as above. This leads then to some observations concerning the John-Nirenberg distribution inequality for $F \circ u, u \in BMO$ and $F \in \text{Lip}\ \alpha$.

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1. Introduction

We will consider the question of when a function w and its reciprocal $1/w$ are in BMO. If we assume that $w : R^n \to R_+$ and consider this question for various spaces X, we obtain distinct results. The answer for $L^p(R^n)$ is that if $w, 1/w \in L^p(R^n)$, then $p = \infty$ while $w, 1/w \in L^\infty$ implies that $w \simeq 1$ which is also equivalent to the fact that $w, 1/w \in A_1$ (for the precise definition of the A_p classes see below). It is known that BMO is the right space to consider in place of L^p as $p \to \infty$ in a number of situations and we will give the answer to this question for BMO in this paper.

The definition of BMO is that $f \in BMO$ if

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx = ||f||_* < +\infty$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x)dx$, and Q is a cube with sides parallel to the coordinate axes. It is important to know that the L^1 norm can be replaced by the L^p norm for $0 < p < \infty$,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p} = ||f||_{*,p} \simeq ||f||_*.$$

We need also to recall the John-Nirenberg lemma, the reason for the above result, for functions of bounded mean oscillation. If $f \in BMO$, there are constants $c_1, c_2 > 0$ independent of f and Q such that

$$|\{t \in Q : |f(t) - f_Q| > \lambda\}| \le c_1 e^{-c_2 \lambda / ||f||_*} |Q|,$$

for all $\lambda > 0$. Of course, bounded functions are in BMO and $\ln 1/|x|$ is an unbounded function in BMO. The precise space we will study is

$$BMO_* = \{w : R^n \to R_+ : w, 1/w \in BMO\}.$$

We need to recall the A_p weights which are defined by the condition

$$A_p(w) = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} < +\infty,$$

where Q is again a cube. The A_p weights solve the problem of characterizing when the Hardy-Littlewood maximal function maps L^p_w into L^p_w, where $Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy$, and the result is

$$\int |Mf(x)|^p w(x)\,dx \le C^p \int |f(x)|^p w(x)\,dx \longleftrightarrow w \in A_p.$$

We will also need to consider $A_1 = \{w | Mw(x) \le Cw(x)\}$, with the smallest such C being denoted $A_1(w)$ and $A_\infty = \bigcup_{p>1} A_p$. Since the A_p constants decrease by Hölder's inequality, we can set $A_\infty(w) = \lim_{p\to\infty} A_p(w)$. We have the set inclusions

$$A_1 \subseteq A_p \subseteq A_q \subseteq A_\infty\,,$$

where $1 \le p \le q \le \infty$. The A_p weights also solve the corresponding problem for the Hilbert transform

$$Hf(x) = \lim_{\epsilon\to 0}\int_{\epsilon<|x-y|<1/\epsilon} \frac{f(y)}{x-y}\,dy.$$

It is known that if $w, 1/w \in A_p$, then $w \in A_2$, and we may limit our study to the case $1 \le p \le 2$ by the inclusion properties of A_p. It is also known that [1, p. 474]

$$w, 1/w \in \bigcap_{p>1} A_p \Longleftrightarrow \ln w \in clos_{BMO} L^\infty. \tag{1}$$

We say that $w \in RH_{p_0}$(reverse Hölder) if

$$\left(\frac{1}{|Q|}\int_Q w^{p_0}\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q w,$$

and we abbreviate by $RH_{p_0}(w)$ the infimum of all such C. We will use the fact, due to Strömberg and Wheeden, that $w \in RH_{p_0}$ if and only if $w^{p_0} \in A_\infty$. An alternate proof of this fact can be found in [3, Lemma 3.1].

2. Preliminary results

Our first result shows that Hölder continuous functions operate on BMO.

Lemma 1: *If F is Hölder continuous of order α, where $0 < \alpha \le 1$ and $f \in BMO$, then $F \circ f \in BMO$ and $||F \circ f||_* \le 2||F||_{Lip\ \alpha}||f||_*^\alpha$.*

Proof. If there is a constant c such that $\frac{1}{|Q|}\int_Q |f(x) - c|dx \le A$, then it is well known that $||f||_* \le 2A$. We compute

$$\left(\frac{1}{|Q|}\int_Q |F(f(x)) - F(f_Q)|^p dx\right)^{1/p} \le \left(\frac{1}{|Q|}||F||^p_{Lip\,\alpha}\int_Q |f(x) - f_Q|^{\alpha p} dx\right)^{1/p}.$$

Thus we obtain with $p = 1/\alpha$, $||F \circ f||_* \leq 2||F||_{Lip\,\alpha}||f||_*^\alpha$.

This has been, at least partially, observed by many people. If $f \in BMO$, then $|f|^\alpha \in BMO$, for $0 < \alpha \leq 1$ and max{f, g} and min{f, g} are in BMO if f, g are in BMO.

We haven't noticed the converse observed, but it is true. If $||F \circ f||_* \leq A||f||_*^\alpha$, then $F \in Lip\,\alpha$. The proof may be found in [2], but as this is not generally available, we give the proof here. Without loss of generality, we may assume $F(0) = 0$ and consider only cubes centered at the origin since BMO is translation invariant. Suppose that $Q = [-\frac{d}{2}, \frac{d}{2}]^n$ and that

$$f(x) = \begin{cases} x_1 & \text{on the double of } Q \\ 0 & \text{outside the double of } Q. \end{cases}$$

One checks that

$$F(f(x)) = \begin{cases} F(x_1) & \text{for } x \in 2Q, \\ 0 & \text{outside the double of } Q. \end{cases}$$

and since $||f||_* \leq ||f||_\infty \leq \frac{d}{2}$, one finds $\frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} |F(x_1) - F_{Q_1}|\,dx_1 \leq Ad^\alpha$, where Q_1 is the one-dimensional cube $[-\frac{d}{2}, \frac{d}{2}]$, and by the Campanato-Meyer theorem [4], this proves the result.

We can use the lemma to show that there is a close connection between BMO and BMO_*.

Theorem 1: *A real valued function u is in $BMO \iff$ there exists a $w \in BMO_*$ such that $u = w - 1/w$ and $||w||_* + ||1/w||_* \simeq ||u||_*$.*

Proof. If u admits the decomposition, it is clear that $u \in BMO$. If we are given a $u \in BMO$, it is easy to see that the equation for w leads to a quadratic equation with a solution of $w = \frac{1}{2}(u + \sqrt{u^2+4})$. The function $F(x) = \frac{1}{2}(x + \sqrt{x^2+4})$ is everywhere differentiable with derivative bounded by 1. By Lemma 1, $w \in BMO$.

Remark. We note that the same proof proves the corresponding result for functions of vanishing mean oscillation, which are defined as is BMO but when the sup is taken over cubes of side r, and the resulting sup goes to 0 as $r \to 0+$.

Another application of Lemma 1 is to the determination of conditions under which the square of a function belongs to BMO. By Lemma 1 with $F(x) = \sqrt{x}$, it follows that such a function belongs to BMO. We show that more is true.

Lemma 2: *If $f = F(u), F \in Lip\ \alpha, u \in BMO$, then*

$$|\{x \in Q : |f(x) - F(u_Q)| > \lambda\}| \leq c_1 e^{-c_2\lambda^{1/\alpha}/||F||_{Lip\ \alpha}^{1/\alpha}||u||_*}|Q|.$$

Proof. Because $u \in BMO$, by the John-Nirenberg lemma, there are constants c_1 and c_2 such that $|\,\{t \in Q : |u(t) - u_Q| > \lambda\}\,| \leq c_1 e^{-c_2\lambda/||u||_*}|Q|$. Hence, since

$$\{t \in Q : |f(t) - F(u_Q)| > \lambda\} \subseteq \left\{t \in Q : |u(t) - u_Q| > \left(\frac{\lambda}{||F||_{Lip\ \alpha}}\right)^{1/\alpha}\right\},$$

we have the inequality

$$|\{t \in Q : |f(t) - F(u_Q)| > \lambda\}| \leq c_1 e^{-c_2(\frac{\lambda}{||F||_{Lip\ \alpha}})^{1/\alpha}/||u||_*}|Q|,$$

which is the desired result.

Corollary 1: *For any* $\epsilon < c_2$,

$$\int_Q \left(e^{\frac{(c_2-\epsilon)|f(x)-F(u_Q)|^{1/\alpha}}{||F||_{Lip\alpha}^{1/\alpha}||u||_*}} - 1 \right) dx \leq c_1 \left(\frac{c_2 - \epsilon}{\epsilon} \right) |Q|.$$

Proof. Let $\phi(x) = e^{Ax^{1/\alpha}} - 1$, which is increasing with $\phi'(x) = \frac{A}{\alpha} x^{1/\alpha - 1} e^{Ax^{1/\alpha}}$. As long as A is positive,

$$\begin{aligned}
\int_Q e^{A|f(x)-F(u_Q)|^{1/\alpha}}\, dx - |Q| &= \frac{A}{\alpha} \int_0^\infty |\{x \in Q : |f(x) - F(u_Q)| > \lambda\}| \lambda^{1/\alpha - 1} e^{A\lambda^{1/\alpha}}\, d\lambda \\
&\leq \frac{A}{\alpha} c_1 \int_0^\infty e^{-\left(\frac{c_2}{||F||_{Lip\ \alpha}^{1/\alpha}||u||_*} - A \right) \lambda^{1/\alpha}} \lambda^{1/\alpha - 1}\, d\lambda |Q|.
\end{aligned}$$

If we choose A less than the fraction, we can use the fact that

$$\frac{1}{\alpha} \int_0^\infty e^{-\epsilon \lambda^{1/\alpha}} \epsilon \lambda^{1/\alpha - 1}\, d\lambda = \int_0^\infty e^{-u}\, du = 1$$

to obtain the above estimate.

If we modify the choice of ϕ slightly by putting $\psi(x) = e^{Ax^{1/\alpha}}$, we see that for $\alpha \leq 1$, ψ is convex and we can apply Jensen's formula to $Q, p = 1, f = |f(x) - F(u_Q)|$ and if we note that

$$|f_Q - F(u_Q)| = \left| \frac{1}{|Q|} \int_Q (f(x) - F(u_Q)) \right| \leq \frac{1}{|Q|} \int_Q |f(x) - F(u_Q)|,$$

we can make the estimate

$$\begin{aligned}
\psi(2|f_Q - F(u_Q)|) &\leq \psi\left(\frac{1}{|Q|} \int_Q 2|f(x) - F(u_Q)| \right) \\
&\leq \frac{1}{|Q|} \int_Q e^{A2^{1/\alpha}|f(x)-F(u_Q)|^{1/\alpha}}.
\end{aligned}$$

We now combine this with Corollary 1 and obtain

$$\begin{aligned}
\int_Q e^{A|f(x)-f_Q|^{1/\alpha}} &= \int_Q e^{A|f(x)-F(u_Q)+F(u_Q)-f_Q|^{1/\alpha}} \\
&\leq \int_Q e^{A2^{1/\alpha}|f(x)-F(u_Q)|^{1/\alpha} + A2^{1/\alpha}|f_Q - F(u_Q)|^{1/\alpha}} \\
&= e^{A2^{1/\alpha}|f_Q - F(u_Q)|^{1/\alpha}} \int_Q e^{A2^{1/\alpha}|f(x)-F(u_Q)|^{1/\alpha}}.
\end{aligned}$$

If we choose $A2^{1/\alpha} = (c_2 - \epsilon)/(||F||_{Lip\ \alpha}^{1/\alpha}||u||_*)$, we can estimate this and

$$\int_Q e^{A|f(x)-f_Q|^{1/\alpha}} \leq \left(c_1 \left(\frac{c_2 - \epsilon}{\epsilon} \right) + 1 \right) |Q| \psi(2|f_Q - F(u_Q)|)$$

by using Corollary 1 and now we apply Jensen's inequality to get

$$\begin{aligned}\int_Q e^{A|f(x)-f_Q|^{1/\alpha}} &\leq \left(c_1\left(\frac{c_2-\epsilon}{\epsilon}\right)+1\right)|Q|\frac{1}{|Q|}\int_Q e^{A2^{1/\alpha}|f(x)-F(u_Q)|^{1/\alpha}} \\ &\leq \left(c_1\left(\frac{c_2-\epsilon}{\epsilon}\right)+1\right)^2|Q|.\end{aligned}$$

We can now state and prove the following.

Theorem 2: *Consider the set of* $f=F(u), u\in BMO, 0<\alpha\leq 1$. *The following two statements are equivalent.*

(i) $F \in Lip\ \alpha$

(ii) there exists $0<c_1,c_2<\infty, 0<A<\infty$, *independent of* $Q, u\in BMO$ *such that*

$$|\{x\in Q: |f(x)-f_Q|>\lambda\}|\leq c_1 e^{-\frac{c_2\lambda^{1/\alpha}}{A||u||_*}}|Q|.$$

and then $A\simeq ||F||^{1/\alpha}_{Lip\ \alpha}$.

Proof. We will first prove that (i) implies (ii). By restricting the range of integration in the inequality derived after Corollary 1, we see that

$$|E_\lambda|\equiv|\{x\in Q:|f(x)-f_Q|>\lambda\}|\leq e^{-A\lambda^{1/\alpha}}\int_Q e^{A|f(x)-f_Q|^{1/\alpha}}\,dx,$$

since $e^{-A\lambda^{1/\alpha}}e^{A|f(x)-f_Q|^{1/\alpha}}>1$ on E_λ. This is the desired result if we choose $\epsilon=\frac{c_2}{2}$ and A as above.

We next show that (ii) implies (i). We first observe that (ii) implies that for some constants $0<c_3,c_4<\infty$

$$\int_Q \exp\left(\frac{c_3|f(x)-f_Q|^{1/\alpha}}{A||u||_*}\right)\leq c_4|Q|.$$

This implies that

$$L_Q\equiv\frac{1}{|Q|}\int_Q\frac{c_3|f(x)-f_Q|^{1/\alpha}}{A||u||_*}\leq c_4.$$

Hölder now gives us, since $1/\alpha\geq 1$,

$$L_Q\geq\left(\frac{1}{|Q|}\int_Q\frac{c_3^\alpha|f(x)-f_Q|}{A^\alpha||u||_*^\alpha}\right)^{1/\alpha}.$$

Hence

$$\frac{1}{|Q|}\int_Q|f(x)-f_Q|\leq CA^\alpha||u||_*^\alpha.$$

The proof is now completed by an application of [2]; see the argument after Lemma 1.

Corollary 2: *If* $b^k\in BMO$, *then*

$$|\{x\in Q:|b(x)-b_Q|>\lambda\}|\leq c_1 e^{-c_2\lambda^k/||b^k||_*}|Q|.$$

Proof. Apply the above theorem with $u(x) = b^k, F(x) = x^{1/k}$ which is Lipschitz continuous of order $1/k$ with Lipschitz constant 1.

Remark. The argument actually shows that if

$$|\{x \in Q : |u(x) - u_Q| > \lambda\}| \le c_1 e^{-c_2 \lambda^k} |Q|,$$

then

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \le (c_1 + 1)^2 e^{-\frac{c_2 2^{1/\alpha}}{||F||_{Lip\ \alpha}^{1/\alpha}} \lambda^{k/\alpha}} |Q|.$$

Our main result connects the behavior of functions in BMO_* with the A_p classes.

Theorem 3: *The set of nonnegative functions which are BMO along with their reciprocals is contained in the intersection of all the A_p classes for $p > 1$, i.e. $BMO_* \subseteq \bigcap_{p>1} A_p$.*

Remarks.(1) Of course, if $b \in BMO_*$, then $1/b \in BMO_* \subseteq \bigcap_{p>1} A_p$ and (1) above implies $\ln\, b \in clos_{BMO} L^\infty$.

(2) The class BMO_* is non-empty. For example, $b_1(x) = \max(\ln 1/|x|, e) \in BMO$ and $1/b_1 \in L^\infty \subseteq BMO$. Moreover, if we take

$$b_2(x) = \max(\ln 1/|x|, 1/\ln(|x|e^2))$$

we get an example of a function which is unbounded and whose inverse is unbounded, yet both $b_2, 1/b_2 \in BMO$.

(3) The result is sharp in the sense that the function b in the theorem cannot be in A_1 since if it were, $1/b$ would also be in A_1 and then by a result of Johnson and Neugebauer [3, Lemma 2.2], $b \simeq 1$.

(4) The converse is, however, not true because with the same function b_1 as above, b_1^2 satisfies $1/b_1^2 \in L^\infty$ and $\ln b_1^2 = 2 \ln b_1 \in clos_{BMO} L^\infty$ and therefore $b_1^2 \in \bigcap_{p>1} A_p$ and $\frac{1}{b_1^2} \in \bigcap_{p>1} A_p$, but $b_1^2 \notin BMO$.

We will prove Theorem 3 as a special case of a more general result, but let us indicate how it can be proved directly. The first step is a lemma.

Lemma 3: *Let us denote by*

$$f_Q = \frac{1}{|Q|} \int_Q f(x)dx,$$

then we have

$$(fg)_Q - f_Q g_Q = \frac{1}{|Q|} \int_Q (f(x) - f_Q)(g(x) - g_Q)dx.$$

Proof. Compute and use the fact that $g - g_Q$ has mean value zero.

We are ready for the first step in this version of the proof of Theorem 3.

Theorem 4: *Suppose $b \in BMO_*$, then b is in A_2.*

Proof. Apply Lemma 3 to b and $1/b$ which gives

$$1 - b_Q(1/b)_Q = \frac{1}{|Q|}\int_Q (b(x) - b_Q)(1/b(x) - (1/b)_Q)dx$$

and allows us to make the estimate $|1 - b_Q(1/b)_Q| \leq ||b||_*||1/b||_*$. Hölder's inequality shows that $1 \leq b_Q(1/b)_Q$ and the above becomes $1 \leq b_Q(1/b)_Q \leq 1 + ||b||_*||1/b||_*$.

Theorem 5: *If $b \in BMO_*$, then $b \in A_{3/2}$.*

For the proof of this statement we have to estimate

$$\left(\frac{1}{|Q|}\int_Q b\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{b^2}\right)^{1/2}.$$

First we require another lemma.

Lemma 4: *With the same notation as in Lemma 3, we have*

$$\frac{1}{|Q|}\int_Q (f(t) - f_Q)(g(t) - g_Q)(h(t) - h_Q)(l(t) - l_Q)dt$$

$$= (fghl)_Q - f_Q(ghl)_Q - g_Q(fhl)_Q - h_Q(fgl)_Q - l_Q(fgh)_Q + f_Qg_Q(hl)_Q$$
$$+ f_Qh_Q(gl)_Q + f_Ql_Q(gh)_Q + g_Qh_Q(fl)_Q + g_Ql_Q(fh)_Q$$
$$+ h_Ql_Q(fg)_Q - 3f_Qg_Qh_Ql_Q.$$

Proof. We expand the integrand and compute the resulting terms.

Take $f = h = b, g = l = \frac{1}{b}$. We obtain

$$1 - b_Q(\frac{1}{b})_Q - (\frac{1}{b})_Q b_Q - b_Q(\frac{1}{b})_Q - (\frac{1}{b})_Q b_Q$$
$$+ \left\{b_Q(\frac{1}{b})_Q + (b_Q)^2(\frac{1}{b^2})_Q + b_Q(\frac{1}{b})_Q + (\frac{1}{b})_Q b_Q + ((\frac{1}{b})_Q)^2(b^2)_Q + b_Q(\frac{1}{b})_Q\right\}$$
$$-3(b_Q)^2((\frac{1}{b})_Q)^2$$
$$= \frac{1}{|Q|}\int_Q (b(t) - b_Q)^2\left(\frac{1}{b}(t) - (\frac{1}{b})_Q\right)^2 dt.$$

This allows us to estimate

$$1 + (b_Q)^2(\frac{1}{b^2})_Q + \left((\frac{1}{b})_Q\right)^2(b^2)_Q - 3(b_Q)^2\left((\frac{1}{b})_Q\right)^2 \leq ||b||^2_{*,4}||\frac{1}{b}||^2_{*,4}$$

which means that

$$1 + (b_Q)^2(\frac{1}{b^2})_Q + \left((\frac{1}{b})_Q\right)^2(b^2)_Q \leq 3(b_Q)^2\left((\frac{1}{b})_Q\right)^2 + ||b||^2_{*,4}||\frac{1}{b}||^2_{*,4}.$$

In particular, $b_Q(\frac{1}{b^2})_Q^{1/2} \leq ||b||_*||\frac{1}{b}||_* + \sqrt{3}A_2(b)$, which proves that

$$A_{3/2}(b) \leq \sqrt{3} + (\sqrt{3} + 1)||b||_*||\frac{1}{b}||_*.$$

The remainder of the direct proof of Theorem 3 proceeds like this. To prove that $b, 1/b$ are in $A_{4/3}$ do the corresponding formula with 8 terms of which 4 are b and 4 are $1/b$, etc. ....

3. A_p weights whose reciprocals are A_p weights

We will now obtain Theorem 3 as a special case of the next result.

Theorem 6: *Suppose* $1 < p_0 \leq 2$. *Then the following are equivalent.*

$$w, 1/w \in A_{p_0} \tag{2}$$

$$L_Q = \frac{1}{|Q|}\int_Q |w - w_Q|^{p_0'-1}|\frac{1}{w} - (\frac{1}{w})_Q|^{p_0'-1} \leq c < +\infty. \tag{3}$$

PROOF. Suppose (2) holds. Let $r = p_0' - 1 \geq 1$. Note that

$$\begin{aligned} L_Q &\leq \frac{1}{|Q|}\int_Q |w^r - (w_Q)^r||\frac{1}{w^r} - (\frac{1}{w})_Q^r| \\ &\leq 1 + (w^r)_Q(\frac{1}{w})_Q^r + w_Q^r(\frac{1}{w^r})_Q + w_Q^r(\frac{1}{w})_Q^r \\ &\leq 1 + A_{p_0}(\frac{1}{w}) + A_{p_0}(w) + A_2(w)^r \leq c < +\infty, \end{aligned}$$

because $w \in A_{p_0}$ implies $w \in A_2$.

Conversely, if (3) holds, then we first note that $w \in A_2$. This follows from the next sequence of inequalities:

$$\begin{aligned} c^{1/r} &\geq L_Q^{1/r} \geq \frac{1}{|Q|}\int_Q |w - w_Q||\frac{1}{w} - (\frac{1}{w})_Q| \\ &\geq \frac{1}{|Q|}\int_Q (w - w_Q)((\frac{1}{w})_Q - \frac{1}{w}) \\ &= w_Q(\frac{1}{w})_Q - 1 - w_Q(\frac{1}{w})_Q + w_Q(\frac{1}{w})_Q. \end{aligned}$$

We use the fact that if $r \geq 1$, then $|a^r - b^r| \geq \frac{a^r}{2^{r-1}} - b^r$. Write

$$\left|(w - w_Q)(\frac{1}{w} - (\frac{1}{w})_Q)\right| = \left|w(\frac{1}{w})_Q + \frac{1}{w}w_Q - (w_Q(\frac{1}{w})_Q + 1)\right|$$

which allows us to estimate the integrand below by

$$\begin{aligned} \left|(w - w_Q)(\frac{1}{w} - (\frac{1}{w})_Q)\right|^r &\geq \frac{1}{2^{r-1}}\left\{w(\frac{1}{w})_Q + \frac{1}{w}w_Q\right\}^r - \left(w_Q(\frac{1}{w})_Q + 1\right)^r, \\ &\geq \frac{1}{2^{r-1}}\left\{w^r(\frac{1}{w})_Q^r + \frac{1}{w^r}w_Q^r\right\} - \left(w_Q(\frac{1}{w})_Q + 1\right)^r. \end{aligned}$$

Now we take the average of this over Q which gives

$$\frac{1}{2^{r-1}}\left\{(w^r)_Q(\frac{1}{w})_Q^r + (\frac{1}{w^r})_Q w_Q^r\right\} \leq c + (A_2(w) + 1)^r,$$

and we conclude that $w, \frac{1}{w} \in A_{p_0}$.

Theorem 3 follows from this result; in fact, we obtain the estimate

$$L_Q \leq ||w||_{*,p(p_0'-1)}^{p_0'-1}||\frac{1}{w}||_{*,p'(p_0'-1)}^{p_0'-1}$$

and as BMO is characterized by $||f||_{*,p}$ for any $p > 0$, we can have any $p_0 > 1$ which proves the result.

Although we proved Theorem 6 for A_p, it immediately implies a result about RH_r.

Theorem 7: *The following statements are equivalent for* $1 \le r < \infty$*:*

$$w, 1/w \in RH_r \tag{4}$$

$$w, 1/w \in A_{1+1/r} \tag{5}$$

$$w^r \in A_2. \tag{6}$$

Proof. (4) $\to$ (5). Since $w^r, 1/w^r \in A_\infty$, we have that $w^r, 1/w^r \in A_2$, and hence $w, 1/w \in A_{1+1/r}$.

(5) $\to$ (4). $w \in A_{1+1/r} \to 1/w \in RH_r$. Similarly, $w \in RH_r$.

(4) $\to$ (6). Since $w^r, 1/w^r \in A_\infty$, we have that $w^r \in A_2$ as above.

(6) $\to$ (4). Since $w^r \in A_2$, $w \in A_{1+1/r} \to 1/w \in RH_r$. From the fact that $w^r \in A_2$, it follows that $w^{-r} \in A_2$ and this implies that we can apply the above remark to $1/w$.

Theorem 8: *Suppose* $u \in BMO$ *and* $\alpha > 0$*. Then* $u^2 + \alpha \in \bigcap_{p>1} A_p$*.*

Proof. For any $\lambda > 0$, write $\lambda u = w_\lambda - 1/w_\lambda$, for some $w_\lambda \in BMO_*$. Then $\lambda^2 u^2 = w_\lambda^2 + \frac{1}{w_\lambda^2} - 2$. By Theorem 7, $w_\lambda^2 \in A_\infty$ and since $w_\lambda \in \bigcap_{p>1} A_p$, by Lemma 2.4 in [3], $w_\lambda^2 \in \bigcap_{p>1} A_p$ and a similar result holds for $\frac{1}{w_\lambda^2}$. This shows that $\lambda^2 u^2 + 2 \in \bigcap_{p>1} A_p$ and hence, $u^2 + \frac{2}{\lambda^2} \in \bigcap_{p>1} A_p$. Since λ is an arbitrary positive number, the result follows.

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