Deriving the Formula for the Sum of a Geometric Series

In Chapter 2, in the section entitled "Making 'cents' out of the plan, by chopping it into chunks", I promise to supply the formula for the sum of a geometric series and the mathematical derivation of it. Here it is.

Consider a sum of terms each of which is a successively higher power of a number or an algebraic quantity represented by a variable:

$$1 + x + x^2 + \ldots + x^n .$$

Note that the first term, that is "1", is also a power, namely x^0, and of course the expression x can also be written x^1. So the geometric series can also be written

$$x^0 + x^1 + x^2 + \ldots + x^n .$$

The word geometric comes from the fact that each term is obtained from the preceding one by *multiplication* by the quantity x. The trick for adding up the series was observed by mathematicians long ago, and is tantamount to nothing more than the process of polynomial multiplication that you encountered in elementary algebra. In fact, consider the following product:

$$(1 - x)(1 + x + x^2 + \ldots + x^n) .$$

The rules of elementary algebra say that to multiply out these polynomials, you multiply the series by "1", then by "x" and then you subtract the two expressions. The result is

$$(1 + x + x^2 + \ldots + x^n) - (x + x^2 + \ldots + x^n + x^{n+1}) .$$

But you should see that there is a tremendous amount of cancellation in the above expression. In fact, all the terms cancel except the first term inside the first parenthesis, and the last term inside the second. That is, all that remains is

$$1 - x^{n+1} .$$

Summarizing we have shown that

$$(1 - x)(1 + x + x^2 + \ldots + x^n) = 1 - x^{n+1} .$$

Finally, dividing through by $1 - x$, we obtain the classic formula for the sum of a geometric series:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} . \qquad \text{(Formula 1)}$$

Now the precise expression that we needed to add up in Chapter 2 was

$$x + x^2 + \ldots + x^n ,$$

that is, the leading term "1" is omitted. Therefore to add that series up, we only need to subtract "1" from Formula 1, to obtain:

$$x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} - 1 = \frac{1 - x^{n+1}}{1 - x} - \frac{1 - x}{1 - x} = \frac{x - x^{n+1}}{1 - x} = x \frac{1 - x^n}{1 - x} . \qquad \text{(Formula 2)}$$

Formulas 1 and 2 are applied several times in the book, usually with $x = 1 + r$ for some interest rate r.