Abstract:
Simulations of turbulent plasma in a strong magnetic field can take advantage of the gyrokinetic approximation, the result of which is a closed set of equations that can be solved numerically. An existing code, GSP, uses a novel and highly efficient solution method to solve the nonlinear 5D gyrokinetic equation. In this project, we will seek to change the velocity-space representation in GSP. This transformation will simplify the inclusion of a new collision operator and make the algorithm more suitable for simulations of turbulence in Tokamak plasmas, while retaining the efficiency and accuracy of the original code.
Outline

• Overview of Gyrokinetics

• Description of GSP

• Changes to Algorithm
Overview

- GSP is a Particle-In-Cell (PIC) code to solve the Gyrokinetic equation to study the evolution of turbulence in highly magnetized plasma.

- Gyrokinetic theory treats the many rapidly circulating charges as charged rings.
  - These rings drift parallel and perpendicular to the background magnetic field.
Overview

• The “particles” in the code are not intended to simulate the physical particles, nor the “gyro-averaged” rings of charge.
  • The evolution of many physical charges are described by a statistical distribution function in phase space: \( f(r, v_\parallel, v_\perp, \theta) \)
  \[
  f \approx F_0 + \delta f + \ldots
  \]
  • Performing an asymptotic expansion and gyroaveraging the Fokker-Planck equation, we can get a dynamical equation for \( \delta f \): the gyrokinetic equation
The Gyrokinetic Equation

\[ \frac{\partial \langle \delta f \rangle}{\partial t} + v_\parallel \frac{\partial}{\partial z} \langle \delta f \rangle + \mathbf{v}_D \cdot \nabla \langle \delta f \rangle = \langle C[\delta f] \rangle - \mathbf{v}_D \cdot \nabla F_0 + v_\parallel \langle E_z \rangle F_0 \]

Where:

- \( \langle \delta f \rangle = \langle \delta f \rangle(R, v_\perp, v_\parallel) \) is the perturbed distribution function to be solved for
- The given background equilibrium distribution is: \( F_0 = n \left( \frac{m}{2\pi kT} \right) e^{-\frac{mv^2}{2kT}} \)
- The particle “weight” is \( w = \frac{\langle \delta f \rangle}{F_0} \)
- The (characteristic) drift velocity is: \( \mathbf{v}_D = \frac{c}{B_0} \langle \mathbf{E} \rangle \times \hat{z} \)
- \( C[\delta f] \) is the collision operator

(angle brackets signify the gyro-averaging operation at constant \( R \))
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The Gyrokinetic Equation

\[
\frac{\partial \langle \delta f \rangle}{\partial t} + v_\parallel \frac{\partial}{\partial z} \langle \delta f \rangle + \mathbf{v}_D \cdot \nabla \langle \delta f \rangle = \langle C[\delta f] \rangle - \mathbf{v}_D \cdot \nabla F_0 + v_\parallel \langle E_z \rangle F_0
\]

To solve this equation, we identify characteristic trajectories in phase space so that:

- The left hand side becomes \( \frac{d}{dt} \langle \delta f \rangle \)
- \( v_\parallel \hat{\mathbf{z}} + \mathbf{v}_D \) is the characteristic velocity
- Characteristic curves are constant in velocity space only under special conditions:
  - Uniform equilibrium magnetic field \( \mathbf{B}_0 \)
  - Special choice of velocity coordinates: \( \mu \equiv \frac{mv_\perp^2}{2B_0} \quad \mathcal{E} = \frac{1}{2} m(v_\perp^2 + v_\parallel^2) \)
GSP Code

- **Step 1: Initialize particles in phase space**
- **Step 2: Predictor Step:**
  - Calculate fields at step $n$
  - Calculate marker weights along characteristics for step $n+1/2$
  - Advance marker positions for half a timestep
  - Update weights with collision operator for step $n+1/2$
- **Step 3: Corrector Step:**
  - Calculate fields at step $n+1/2$
  - Calculate marker weights along characteristics for step $n+1$
  - Advance marker positions for a full timestep
  - Update weights with collision operator for step $n+1$
- **Step 4: Output results**
- Repeat Steps 2-4 $N_T$ times
Flowchart

- Calculate Fields
- Update Weights
- Advance Markers
- Apply Collision Operator
- Calculate Fields
Flowchart

- **Calculate Fields**
- **Update Weights**
- **Advance Markers**
- **Apply Collision Operator**
Calculate Fields

\[ \mathbf{v}_D = \frac{c}{B_0} \langle \mathbf{E} \rangle \times \hat{z} \]

- Once we know the electrostatic potential, we know the electric field:
  \[ \mathbf{E} = -\nabla \phi \]

- The gyro-averaging operation is simplified in Fourier space:
  \[ \langle \tilde{\mathbf{E}} \rangle = J_0 \left( \frac{k_{\perp} v_{\perp}}{\Omega} \right) \tilde{\mathbf{E}} = -i k \tilde{\phi} J_0 \left( \frac{k_{\perp} v_{\perp}}{\Omega} \right) \]

\( J_0 \) is the 0-th order Bessel function

\[ \Omega = \frac{eB_0}{mc} \]
Calculate Fields

- First, deposit charges onto grid in $\mathbb{R}$-space
- Fourier transform grid
- Apply Poisson’s equation to calculate the potential:

$$\tilde{\phi} \propto \int dv_{\parallel} \int v_{\perp} dv_{\perp} \left\langle \delta f \right\rangle_{R} J_{0} \left( \frac{k_{\perp} v_{\perp}}{\Omega} \right)$$

- Issue:
  - $J_{0}$ is expensive to calculate explicitly every time

- Solution:
  - Discretize $v_{\perp}$ on a grid in velocity space
  - Store $J_{0}$ in a table at the relevant discrete values of $v_{\perp}$ and $k_{\perp}$
Calculate Fields

- Once potential is calculated, find the gyro-averaged fields in k-space

\[ \langle \tilde{E} \rangle_R = -i k \tilde{\phi} J_0 \left( \frac{k_{\perp} v_{\perp}}{\Omega} \right) \]

- Fourier transform back into grid in R-space

- Interpolate to find fields at marker positions
Flowchart

- Calculate Fields
- Update Weights
- Advance Markers
- Apply Collision Operator
Update Weights and Advance Markers

• This Monte Carlo scheme [Aydemir, 1994] doesn’t model $\langle \partial f \rangle_R$ explicitly, but rather the function:

$$w = \frac{\langle \partial f \rangle_R}{F_0}$$

• The equation for $w$ along characteristics is:

$$\dot{w} = \frac{q}{T} v_{\parallel} \langle E_z \rangle_R - \frac{\nabla F_0}{F_0} \cdot \mathbf{v}_D$$

• Update marker positions explicitly using $v_{\parallel} \hat{z} + \mathbf{v}_D$
Flowchart

1. Calculate Fields
2. Update Weights
3. Advance Markers
4. Apply Collision Operator

Flowchart Diagram:
- Calculate Fields
- Update Weights
- Advance Markers
- Apply Collision Operator
Collision Operator

• Pitch-angle scattering operator
  • Diffusive in velocity space
  • Energy is conserved with like-particle interactions, or when there’s a large disparity in mass (such as ions and electrons)
  • $\nu$ is a constant parameter

• Collision operator defined in terms of $h = \langle \partial f \rangle + \langle \phi \rangle F_0$

$$C[h] = \nu \frac{\partial}{\partial \xi} \left( 1 - \xi^2 \right) \frac{\partial h}{\partial \xi} \quad \xi = \frac{v_\parallel}{\nu}$$
Collision Operator

• Godunov splitting: We have already applied the non-collisional part of the derivative:

\[ \langle \delta f \rangle^n \rightarrow \langle \delta f \rangle^* \]

• First, convert \( \delta f^* \) to \( h^* \):

\[ h^* \equiv \langle \delta f \rangle^* + \langle \phi \rangle F_0 \]

• Find the derivatives implicitly

\[ \frac{h^{n+1} - h^*}{\Delta t} = C[h^{n+1}] \]

• Invert the tri-diagonal matrix to obtain \( h^{n+1} \)

\[ \langle \delta f \rangle^{n+1} = h^{n+1} - \langle \phi \rangle F \]
Collision Operator

• Issues:
  • As implemented, the collision operator does not obey the proper
    conservations laws

• The update will use the operator from [Abel et al, 2008], which
  conserves particles, momentum, energy, and obeys the Boltzmann
  H-theorem
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Summary of Changes

- **Calculate Fields**: Most difficult task. Perform calculation with new velocity-space coordinates. Try to keep repeated “calculation” of $J_0$ efficient.

- **Update Weights Advance Markers**: Minor changes if any

- **Apply Collision Operator**: Rework – convert pitch-angle operator to new coordinates. Apply Abel operator.
Coding Status

- **Calculate Fields**: In progress
- **Update Weights**
  - **Advance Markers**: No changes needed
- **Apply Collision Operator**
Integral for Electrostatic Potential

\[ \tilde{\phi} \propto \int dv_\parallel \int v_\perp dv_\perp \langle \delta f \rangle_R J_0 \left( \frac{k_\perp v_\perp}{\Omega} \right) \]

- We’re representing the distribution as:

\[ \langle \delta f \rangle_R = F_0 w(r, v_\parallel, v_\perp) = F_0 \sum_i w_i \delta(r - R_i) \delta(v_\parallel - v_{\parallel,i}) \delta(v_\perp - v_{\perp,i}) \]

so to calculate the integral, we just sum up the values of \( J_0 \cdot w \) for each particle and normalize

- When \( v_\perp \) is allowed to change, we need another way to looking up \( J_0 \)
Current Velocity Grid

- Particles interpolated onto \( \mathbf{k} \) grid
- Particles already have an assigned \( v_\perp \)

\[
J_0 = J_0(k_x, k_y, v_\perp)
\]
New Velocity “Grid”

- Particles scattered in 2D velocity space
- Can move around in velocity space depending on the local magnetic field $B(z)$

$$J_0 = J_0(k_x, k_y, v_\perp, B)$$

$$= J_0\left(k_x, k_y, \frac{\mu}{B}\right)$$
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Summary

- Upgrade to GSP progressing slightly behind schedule
  - Phase I (Gyrokinetics analytics and algorithm understanding) took longer than initially expected
  - Currently on Phase II

- Coding still on track to be completed before February
Questions?
Gyrokinetic Derivation Summary

- Start with Fokker-Planck equation

- Gyrokinetic ordering assumptions:
  \[
  \frac{\omega}{\Omega} \sim \frac{v}{\Omega} \sim \frac{k_{||}}{k_{\perp}} \sim \frac{v_D}{v_t} \sim \frac{\delta f}{F_0} \sim \epsilon
  \]

- Order $\epsilon^0$:
  - $F_0$ independent of gyroangle $\theta$

- Order $\epsilon^1$:
  - $F_0$ Maxwellian
  - $\delta f = -\frac{q\phi}{T} F_0 + h$ where $\frac{\partial h}{\partial \theta} = 0$

- Order $\epsilon^2$:
  - The gyrokinetic equation
Gyroaveraging Fourier Transforms

\[
\left< e^{i \mathbf{k} \cdot \mathbf{r}} \right>_R = \frac{1}{2\pi} \int e^{i \mathbf{k} \cdot (\mathbf{R} + \rho)} d\theta = \frac{1}{2\pi} e^{i \mathbf{k} \cdot \mathbf{R}} \int e^{-i \frac{\mathbf{k} \cdot \mathbf{v}}{\Omega}} d\theta
\]

\[
= \frac{1}{2\pi} e^{i \mathbf{k} \cdot \mathbf{R}} \int e^{-i \frac{k_{\perp}}{\Omega} \cos \theta} d\theta
\]

\[
= e^{i \mathbf{k} \cdot \mathbf{R}} J_0 \left( \frac{k_{\perp}}{\Omega} \right)
\]
Maxwell’s Equations

- **Poisson’s Equation:** \( \nabla^2 \phi = -4\pi \left( q_i n_i + q_e n_e \right) \)

- **Ampere’s Law:** \( \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J} \)

\[ n_i = \int \delta f_i(r, v, t) d^3v \]
\[ n_e = n_0 \frac{e\phi}{T_e} \]
\[ \mathbf{J} = e n_i \mathbf{v}_i - e n_e \mathbf{v}_e \]
\[ n_i \mathbf{v}_i = \int \delta f_i v d^3v \]
Update Weights and Advance Markers

- Now advance markers along characteristic trajectories:

  \[
  x_i = x_i + \Delta t (\hat{x} \cdot v_{Di}) \\
  y_i = y_i + \Delta t (\hat{y} \cdot v_{Di}) \\
  z_i = z_i + \Delta t (v_{\parallel i})
  \]

- The markers’ velocity space coordinates do not need updating.
- As currently implemented, markers have \( v_{\parallel} \) and \( v_{\perp} \) assigned, which are constants of motion *only in a uniform magnetic field*.
  - This update will change coordaintes to \( E \) and \( \mu \) which remain constants of motion (up to the order required) even when the magnetic field is non-uniform.