Solving the steady state diffusion equation with uncertainty

Mid-year presentation

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Problem

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$$-\nabla \cdot (k(x, \omega)\nabla u) = f(x) \; , \tag{1}$$

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- The boundary conditions are deterministic.

$$u(x,\omega) = g(x) \text{ on } \partial D_D$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D_n \ .$$

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Validation

- Compare the moments of this solution to the moments obtained from solving the equation using the Monte-Carlo method.

Karhunen-Loéve expansion

The expansion is

$$a(x,\vec{\xi}) = \mu(x) + \sum_{s=1}^{\infty} \sqrt{\lambda_s} f_s(x) \xi_s \ . \tag{2}$$

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- The random variables ξ_s are uncorrelated.
- The λ_s and $f_s(x)$ are eigenpairs which satisfy

$$(\mathcal{C}f)(x) = \int_D C(x,y) f(y) dy = \lambda f(x) \ , \tag{3}$$

where $C(x,y)$ is the covariance function of the random field.

Covariance matrix

The covariance matrix for a finite set of points x_i in the spatial domain is

$$C(x_i, x_j) = \int_\Omega (a(x_i, \omega) - \mu(x_i))(a(x_j, \omega) - \mu(x_j)) dP(\omega) , \tag{4}$$

where

$$\mu(x) = \int_\Omega a(x, \omega) dP(\omega) . \tag{5}$$

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Denote the approximation to this matrix

$$C_{ij} = C(x_i, x_j) \ . \qquad (6)$$

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- For a uniform two-dimensional domain with interval sizes h_x and h_y, the problem to solve is

$$h_x h_y CV = \Lambda V \ . \tag{8}$$

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- Otherwise, n samples can be taken at each spatial point to form the sample covariance matrix, $\widehat{C}$.

$$\widehat{C}_{ij} = \frac{1}{n}\sum_{k=1}^{n}(a(x_i, \xi_k) - \hat{\mu}_i)(a(x_j, \xi_k) - \hat{\mu}_j) \tag{9}$$

$$\hat{\mu}_i = \frac{1}{n}\sum_{k=1}^{n} a(x_i, \xi_k) \ . \tag{10}$$

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- Define a matrix:

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- Then the sample covariance matrix can be written as

$$\widehat{C} = \frac{1}{n} B B^T \, . \tag{12}$$

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- The eigenvalues of $\widehat{C}$ will be $\frac{1}{n}\Sigma^2$.
- The eigenvectors of $\widehat{C}$ will be the columns of U.
- Using this approach ensures that small numerical errors will not produce imaginary eigenvalues.

Gaussian random field

- A Gaussian random field in one dimension has covariance function

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- σ^2 is the (constant) variance of the stationary random field and b is the correlation length.
- Large values of b: random variables at points that are near each other are highly correlated.

Gaussian random field

- Exact solutions for the eigenvalues and eigenfunctions are known [9].

$$\lambda_n = \sigma^2 \frac{2b}{\omega_n^2 + b^2} \tag{14}$$

$$\lambda_n^* = \sigma^2 \frac{2b}{\omega_n^{*2} + b^2} \tag{15}$$

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where ω_n and ω_n^* solve the following:

$$b - \omega \tan(\omega a) = 0 \tag{16}$$

$$\omega^* + b \tan(\omega^* a) = 0 \ . \tag{17}$$

Gaussian random field

- The random variables in the expansion are $\xi_s \sim N(0,1)$.

$$a(x,\vec{\xi}) = \mu(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} f_n(x) \xi_n \tag{18}$$

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- For a two-dimensional Gaussian field

$$C((x_1,y_1),(x_2,y_2)) = \sigma^2 \exp\left(\frac{-|x_1 - x_2|}{b_1} - \frac{-|y_1 - y_2|}{b_2} \right) \tag{19}$$

Verification for 1D Gaussian random field

Three methods were used to find the eigenvalues of a one-dimensional $N(0,1)$ Gaussian random field on $D = [-1, 1]$ with step size $h = .02$.

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Implemented using Matlab and made use of functions written by E. Ullman 2007-10.

Gaussian random field 1D

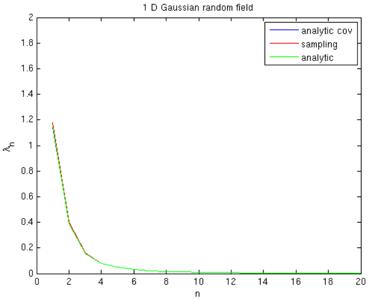
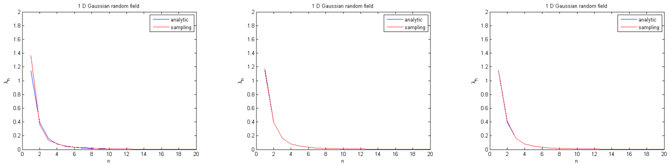


Figure: Eigenvalues of Gaussian random field with parameters $b = 1$, $n = 10000$ for the three methods. Methods 1 and 2 produce nearly identical results.

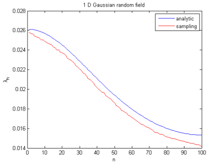
Gaussian random field 1D



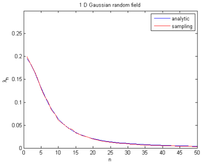
(a) n=100 (b) n=1000 (c) n=10000

Figure: The eigenvalues of the sampling method converge as the number of samples, n is increased.

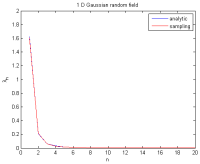
Gaussian random field 1D



(a) b = 0.01, n=100000



(b) b = 0.1, n=1000



(c) b = 3, n=10000

Figure: The effect of correlation length, b, on the eigenvalues

Gaussian random field

- Verified three methods using a two-dimensional domain $D = [0, 1]x[0, 1]$ as well.
- Eigenvectors also agree.

Lognormal random field

- If $a(x, \xi)$ is a Gaussian random variable, $k(x, \xi) = \exp(a(x, \xi))$ is lognormal at every point in the spatial domain.

Lognormal random field

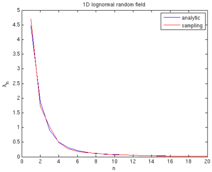
- If $a(x,\xi)$ is a Gaussian random variable, $k(x,\xi) = \exp(a(x,\xi))$ is lognormal at every point in the spatial domain.
- If $X \sim N(\mu,\sigma)$ and $X = \ln(Y)$, the lognormal random variable Y has the following results [10]:

$$E[Y] = e^{\sigma^2/2} \tag{20}$$

$$Var[Y] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \tag{21}$$

$$LC(x_1,x_2) = e^{2\mu+\sigma^2}(e^{C(x_1,x_2)} - 1)\,. \tag{22}$$

Lognormal random field 1D



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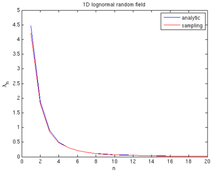
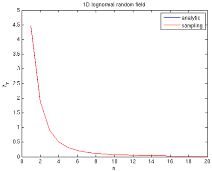


(b) n=1000



(c) n=10000

Figure: The eigenvalues obtained using the sample covariance matrix, converge to the analytic covariance matrix results as the number of samples is increased. Tests use correlation length $b = 1$.

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$$k(x, \vec{\eta}) = \mu(x) + \sum_{s=1}^{\infty} \sqrt{\lambda_s} f_s(x) \eta_s \ . \tag{23}$$

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- What is the distribution of the η_s?

Schedule

Stage 2: December

- Finish construction of the principal components analysis
- Write code which generates Monte-Carlo solutions

Stage 3: January- February

- Run the Monte-Carlo simulations
- Write solution method

Stage 4: March - April

- Run numerical method
- Analyze accuracy and validity of the method

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