MRI Reconstruction via Fourier Frames on Interleaving Spirals

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Talk Outline

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  - Spectral Representation
  - Problem Construction
  - Matrix Representation
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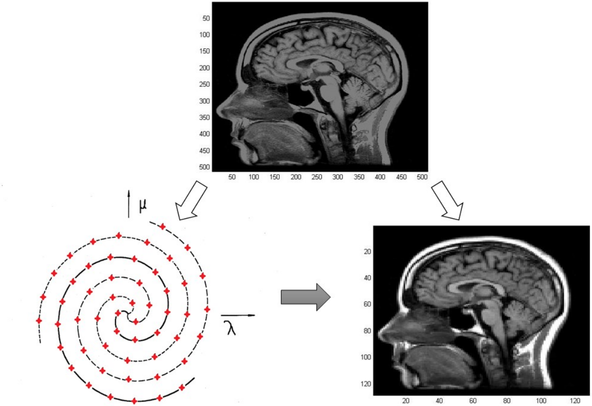
Motivation

Goal: Recover an image using spectral data from an MRI machine sampled along interleaving spirals.

- Spiral sampling makes for faster data acquisition than rectilinear sampling.
- Standard reconstruction over spiral samples requires interpolation.
- We use Beurling's theorem to prove that spiral sampling gives rise to a Fourier frame over the signal space of the image.
- We strive for comparable reconstructions that will ultimately reduce processing time in obtaining MRI images.

Problem Overview

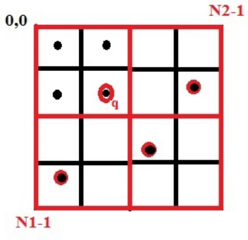
We aim to recover an image $f : E \to \mathbb{R}$ where $E \subset \mathbb{R}^2$ using spectral data from an MRI machine.



Images courtesy of U.S. Patent US5485086 A and “Carolyn’s MRI”, by ClintJCL (Flickr)

Discretization

- High Resolution
  - Let $\chi_1 = \{\square^1_k, p_k \in \square^1_k \ \forall k\}_{k=0}^{B-1}$ be a refined tagged partition of E.
  - Approximate f using the piecewise constant function f_{χ_1}.
- Low Resolution
  - Let $\chi_2 = \{\square^2_j, q_j \in \square^2_j \ \forall j\}_{j=0}^{N_1 N_2 - 1}$ be a coarse tagged partition of E such that f_{χ_2} is piecewise constant.
  - For each $q_j \in \chi_2$, there is a corresponding $p_k \in \chi_1$.



Spectral Representation

Choose $M \geq N_1 N_2$ points $\alpha_i = (\lambda_i, \mu_i) \in \Omega \subset \widehat{\mathbb{R}}^2$ on the interleaving spirals. At the points α_i under the partition χ_1, we have the spectral representation

$$\widehat{f}_{\chi_1}(\alpha_i) = \sum_{k=0}^{B-1} f(p_k)\widehat{\mathbb{1}}_{\square_k^1}(\alpha_i). \qquad (1)$$

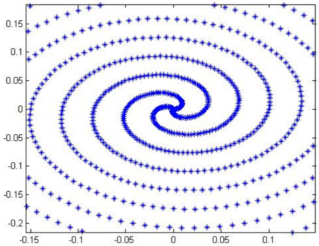


Figure: Points in the frame for 3 interleaving spirals. Limited domain Ω will induce error in the reconstruction.

Problem Construction

Let

$$g = \sum_{j=0}^{N_1 N_2 - 1} c_j \mathbb{1}_{\square_j^2} \tag{2}$$

be an image formed over the coarse partition χ_2. Given the spectral data $\widehat{f}_{\chi_1}(\alpha_i)$, we want to find c_j that satisfy

$$\min_c \sum_{i=0}^{M-1} |\widehat{f}_{\chi_1}(\alpha_i) - \widehat{g}(\alpha_i)|^2 \tag{3}$$

We compare the actual recovered image g to the ideal recovered image f_{χ_2} formed by local averaging over f_{χ_1}.

Matrix Representation

Let

$$\widehat{\mathbb{F}} = [\widehat{f}_{\chi_1}(\alpha_0)\ \widehat{f}_{\chi_1}(\alpha_1)\ \ldots\ \widehat{f}_{\chi_1}(\alpha_{M-1})]^{\mathrm{T}}$$

and

$$\mathbb{F} = [c_0\ c_1\ \ldots\ c_{N_1 N_2 - 1}]^{\mathrm{T}}.$$

Define $\mathbb{H}$ such that $[\mathbb{H}]_{i,j} = H_j(\alpha_i) = \widehat{\mathbb{1}}_{\square_j^2}(\alpha_i)$, where

$$H_j(\alpha_i) = H_j(\lambda_i, \mu_i) = \frac{1}{N_1}\frac{1}{N_2}\mathrm{sinc}\left(\frac{1}{N_1}\lambda_i\right)\ \mathrm{sinc}\left(\frac{1}{N_2}\mu_i\right) e^{-2\pi I(T_n\lambda_i + T_m\mu_i)}$$

We will solve the least-squares problem

$$\mathbb{F} = (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{\mathbb{F}}, \tag{4}$$

$$(N_1 N_2 \times 1) = (N_1 N_2 \times N_1 N_2)(N_1 N_2 \times M)(M \times 1)$$

Transpose Reduction

Define $V_i = (H_0(\alpha_i), ..., H_{N_1 N_2 - 1}(\alpha_i))^*$ such that

$$\mathbb{H} = \begin{pmatrix} H_0(\alpha_0) & \cdots & H_{N_1 N_2-1}(\alpha_0) \\ H_0(\alpha_1) & \cdots & H_{N_1 N_2-1}(\alpha_1) \\ \vdots & \vdots & \vdots \\ H_0(\alpha_{M-1}) & \cdots & H_{N_1 N_2-1}(\alpha_{M-1}) \end{pmatrix} = \begin{pmatrix} V_0^* \\ V_1^* \\ \vdots \\ V_{M-1}^* \end{pmatrix}.$$

Then,

$$\mathbb{H}^*\mathbb{H} = \sum_{i=0}^{M-1} V_i V_i^* \tag{5}$$

and

$$\mathbb{H}^*\widehat{\mathbb{F}} = \sum_{i=0}^{M-1} \widehat{\mathbb{F}}_i V_i. \tag{6}$$

Error Measures

Results are evaluated using the peak signal-to-noise ratio (PSNR) and the structural similarity index (SSIM).

- PSNR, measured in dB, is based on the mean-square error, where $max_{f_{\chi_2}}$ is the maximum possible pixel value for f_{χ_2}.

$$\begin{aligned} \text{MSE} &= \frac{1}{N_1 N_2} \sum_{j=0}^{N_1 N_2 - 1} (g(c_j) - f_{\chi_2}(q_j))^2 & (7) \\ \text{PSNR} &= 10 \log \frac{max_{f_{\chi_2}}^2}{MSE} & (8) \end{aligned}$$

- SSIM estimates luminence and contrast from the mean and standard deviations of the signals. C_1 and C_2 are used as stabilizers.

$$\text{SSIM}(x, y) = \frac{(2\mu_x \mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}. \quad (9)$$

Validation: Uniform Grid

Conjugate gradient (CG) requires 10 iterations for convergence within relative tolerance

$$||\mathbb{H}^*\widehat{\mathbb{F}} - \mathbb{H}^*\mathbb{H}x||_2 < 1e^{-8}||\mathbb{H}^*\widehat{\mathbb{F}}||_2$$

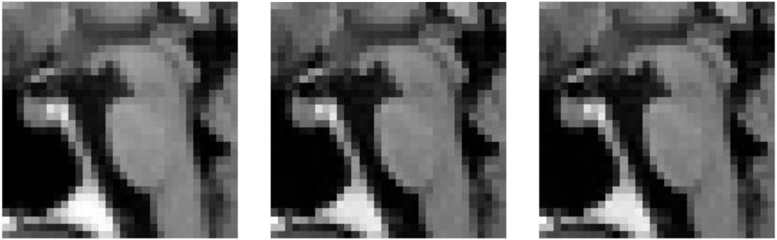


Figure: Uniform grid reconstruction along integers in the domain [-floor($\sqrt{M}$)/2, floor($\sqrt{M}$)/2]^2, $M = 2048$ (oversampling factor of 2). Ideal reconstruction (L), reconstruction via LDL decomposition (M), reconstruction via CG method (R), $N_1 = N_2 = 32$. PSNR: 29.12 dB, SSIM: 0.9895.

Validation: 2x2 Recovery Using Spiral Sampling

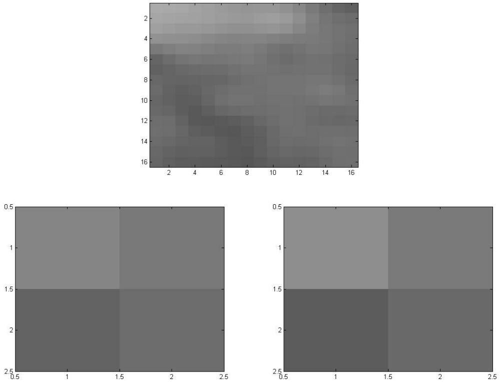


Figure: 2x2 reconstruction test along the spiral with $M = 8$ points in the frame (oversampling factor of 2), $\theta = 0.01$, $m = 3$ interleaving spirals. Top: 16x16 high-resolution image. Left: Ideal image. Right: Recovered image. $N_1 = N_2 = 2$. PSNR: 33.3959 dB, SSIM: 0.9454.

Testing: Oversampling Factor

The number of samples chosen in each reconstruction is $M = K \text{x} N_1 \text{x} N_2$, where $K \in \mathbb{N}$ is the oversampling factor.

Rec. image size (N_1xN_2)	K	PSNR (dB)	SSIM	Avg. Err. Per Pixel
8x8	16	17.8413	0.8035	13.9844
8x8	32	26.9622	0.9690	4.0312
16x16	16	29.8072	0.9900	6.1992
32x32	4	8.6455	0.0131	69.5225
32x32	8	27.3753	0.9875	9.2764

Here,

$$\text{Avg. err. per pixel} = \frac{1}{N_1 N_2} \sum_{j=0}^{N_1 N_2 - 1} |f_{\chi_2}(q_j) - g(c_j)|$$

Testing: Oversampling Factor

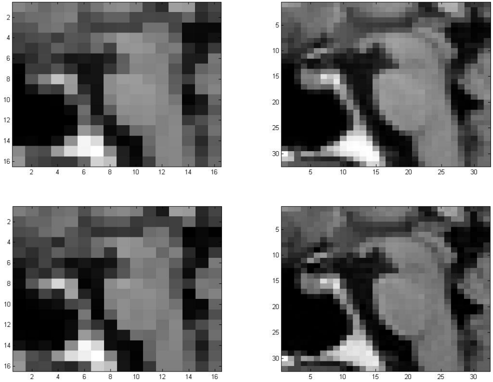


Figure: Spiral reconstruction test from 128x128 high-resolution image, $\theta = 0.01$, $m = 3$. Top: Ideal reconstructions of a 16x16 and 32x32 image. Bottom: Images recovered with different oversampling factor K. L: Num. samples $M = 16 \times 16^2$ ($K = 16$), PSNR: 29.81 dB, SSIM: 0.99. R: Num. samples $M = 8 \times 32^2$ ($K = 8$), PSNR: 27.38 dB, SSIM: 0.98.

Testing: Oversampling Factor

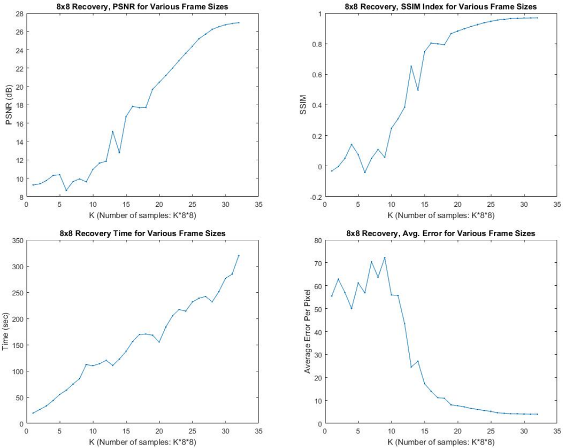


Figure: Reconstruction test varying oversampling factor K, $\theta = 0.01$, $m = 3$. High-resolution image size 128x128. Clockwise from top left: PSNR, SSIM, average error per pixel and run time for varying K in N_1xN_2=8x8 low-resolution reconstruction.

Testing: Spiral Angle

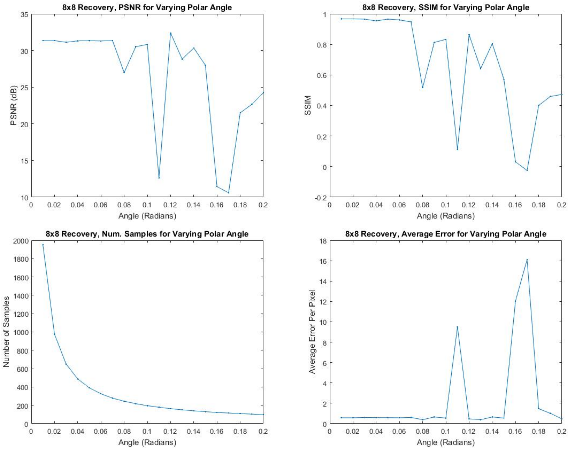


Figure: Reconstruction test varying polar angle θ, $m = 3$. High-resolution image size 128x128. Clockwise from top left: PSNR, SSIM, average error per pixel and number of samples for varying θ in N_1xN_2=8x8 low-resolution reconstruction.

Testing: Method Comparison

For an *n*x*n* matrix, LDL decomposition requires $\frac{n^3}{3}$ flops, while the conjugate gradient method requires n^2 flops.

Image size (N_1xN_2)	cond($\mathbb{H}^*\mathbb{H}$)	LDL time (s)	CG time (s)	Num. CG iter.
2x2	4.33	9.64e-4	2.265e-3	4
4x4	11.40	1.55e-4	1.132e-3	16
8x8	24.94	4.10e-4	1.484e-2	26
16x16	51.43	2.45e-2	7.655e-3	39
32x32	103.78	6.00e-1	1.441e-1	51

Testing: Oasis Database

- Full data set has 416 subjects aged 18 to 96, 100 of whom have varying levels of Alzheimer's disease. Tested subset contains 39 subjects.
- Images are averages of four scans per subject with postprocessing.
- Twenty-two females, seventeen males.
- Nine subjects have very mild dementia, three subjects have mild dementia.

Oasis is provided by Washington University Alzheimers Disease Research Center, the Howard Hughes Medical Institute (HHMI), the Neuroinformatics Research Group (NRG), and the Biomedical Informatics Research Network (BIRN).

Testing: Oasis Database

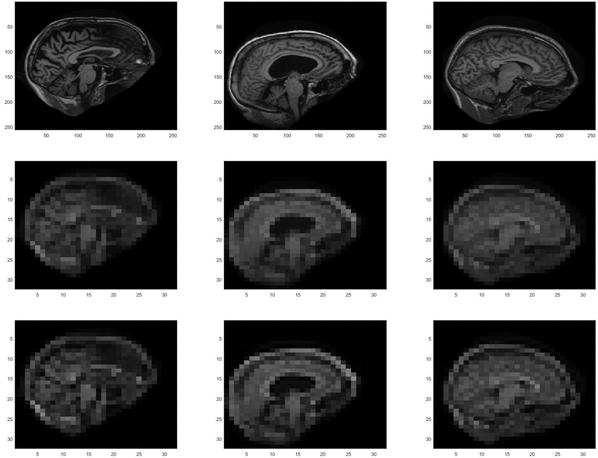


Figure: Oasis reconstruction test, $K = 8$, $\theta = 0.01$, $m = 3$. T: 256x256 high-resolution images. M: Ideal images. B: Recovered images, N_1xN_2=8x8. PSNR: 37.39 dB, 35.96 dB, 37.53 dB. SSIM: 0.9862, 0.9862, 0.9838.

Conclusions

- We can effectively reconstruct an MRI image from spectral data on interleaving spirals.
- The methods are robust across multiple images.
- The oversampling factor K and the polar angle θ must be chosen in a reasonable range.
- The number of samples needed for reconstruction increases with the problem size.
- For the given error tolerance, the CG method and LDL decomposition return the same results. As the problem size increases, CG shows faster convergence.
- Parallelization makes this method feasible in a real-life setting. Basic parallelization decreased the timing overhead by a factor of four.

Timeline

- October 2015: Code the sampling routine to form the Fourier frame. **[Complete]**
- November 2015: Validation on small problems. **[Complete]**
- December 2015: Code the transpose reduction algorithm and begin testing. **[Complete]**
- January 2016: Code the conjugate gradient algorithm (standard and modified). **[Complete]**
- February - March 2016: Error analysis/testing. **[Complete]**
- April 2016: Finalize results. **[Complete]**

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Thank you!

Questions?