MRI Reconstruction via Fourier Frames on Interleaving Spirals

Mid-Year Presentation

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Talk Outline

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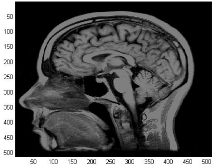
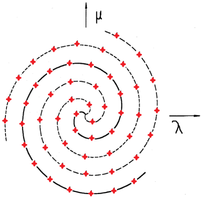
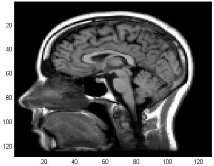
Problem Overview

Goal: Recover an image $f : E \to \mathbb{R}$ where $E \subset \mathbb{R}^2$ using spectral data from an MRI machine.



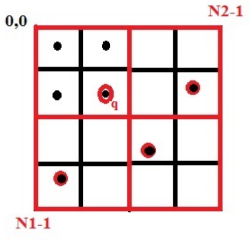




Images courtesy of U.S. Patent US5485086 A and "Carolyn's MRI", by ClintJCL (Flickr)

Discretization

- High Resolution
  - Let $\chi_1 = \{\square_k^1, p_k \in \square_k^1 \; \forall k\}_{k=0}^{B-1}$ be a refined tagged partition of E.
  - We approximate f using the piecewise constant function f_{χ_1} due to lack of access to real MRI data.
- Low Resolution
  - Let $\chi_2 = \{\square_j^2, q_j \in \square_j^2 \; \forall j\}_{j=0}^{N_1 N_2 - 1}$ be a coarse tagged partition of E such that f_{χ_2} is piecewise constant.
  - For each $q_j \in \chi_2$, there is a corresponding $p_k \in \chi_1$.
  - We will approximate f_{χ_2} from f_{χ_1}.



Partition Representation

Under the partition χ_1, we have the representation

$$f_{\chi_1} = \sum_{k=0}^{B-1} f(p_k) \mathbb{1}_{\square_k^1} \tag{1}$$

and the equivalent spectral representation

$$\widehat{f}_{\chi_1} = \sum_{k=0}^{B-1} f(p_k) \widehat{\mathbb{1}}_{\square_k^1}. \tag{2}$$

Generating Spectral Data

Choose $M \geq N_1 N_2$ points $\alpha_i = (\lambda_i, \mu_i) \in \Omega \subset \widehat{\mathbb{R}}^2$ on the interleaving spirals to get

$$\widehat{f}_{\chi_1}(\alpha_i) = \sum_{k=0}^{B-1} f(p_k)\widehat{\mathbb{1}}_{\square_k^1}(\alpha_i). \tag{3}$$

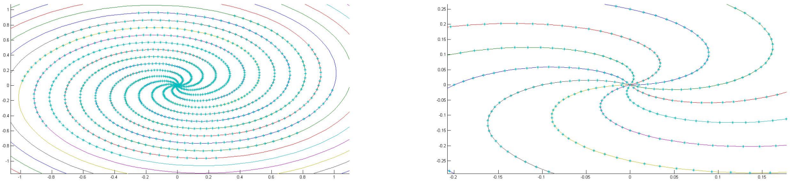


Figure: Points in the frame for 10 interleaving spirals. Limited domain Ω will induce error in the reconstruction.

Spectral Equivalence

Let

$$g = \sum_{j=0}^{N_1 N_2 - 1} c_j \mathbb{1}_{\square_j^2} \tag{4}$$

be an image formed over the coarse partition χ_2. Given the spectral data $\widehat{f}_{\chi_1}(\alpha_i)$, we want to find c_j that solve

$$\min_c \sum_{i=0}^{M-1} |\widehat{f}_{\chi_1}(\alpha_i) - \widehat{g}(\alpha_i)|^2 \tag{5}$$

We compare the actual recovered image g to the ideal recovered image f_{χ_2} formed by local averaging over f_{χ_1}.

Matrix Representation

Let

$$\widehat{\mathbb{F}} = [\widehat{f}_{\chi_1}(\alpha_0)\ \widehat{f}_{\chi_1}(\alpha_1)\ \dots\ \widehat{f}_{\chi_1}(\alpha_{M-1})]^{\mathrm{T}}$$

and

$$\mathbb{F} = [c_0\ c_1\ \dots\ c_{N_1 N_2 - 1}]^{\mathrm{T}}.$$

Define $\mathbb{H}$ such that $[\mathbb{H}]_{i,j} = H_j(\alpha_i)$, where $H_j(\alpha_i) = \widehat{\mathbb{1}}_{\Box_j^2}(\alpha_i)$.

We form the overdetermined system

$$\widehat{\mathbb{F}} = \mathbb{H}\mathbb{F}. \tag{6}$$

$$(M \times 1) = (M \times N_1 N_2)(N_1 N_2 \times 1)$$

Implementation

We will solve the least-squares problem

$$\mathbb{F} = (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{\mathbb{F}}, \tag{7}$$

$$(N_1 N_2 \times 1) = (N_1 N_2 \times N_1 N_2)(N_1 N_2 \times M)(M \times 1)$$

where

- $\mathbb{H}$ is the Bessel map $\ell^2(\{0, 1, ..., N_1 N_2 - 1\}) \to \ell^2(\{0, 1, ..., M - 1\})$
- $\mathbb{H}^*$ is its adjoint
- $\mathbb{H}^*\mathbb{H}$ is the frame operator

We begin with transpose reduction.

Transpose Reduction

Let $A = \mathbb{H}^*\mathbb{H}$ and $b = \mathbb{H}^*\widehat{f}$. Define $V_i = (H_0(\alpha_i), ..., H_{N_1N_2-1}(\alpha_i))^*$ such that

$$\mathbb{H} = \begin{pmatrix} H_0(\alpha_0) & \cdots & H_{N_1N_2-1}(\alpha_0) \\ H_0(\alpha_1) & \cdots & H_{N_1N_2-1}(\alpha_1) \\ \vdots & \vdots & \vdots \\ H_0(\alpha_{M-1}) & \cdots & H_{N_1N_2-1}(\alpha_{M-1}) \end{pmatrix} = \begin{pmatrix} V_0^* \\ V_1^* \\ \vdots \\ V_{M-1}^* \end{pmatrix}.$$

Then,

$$b = \mathbb{H}^*\widehat{\mathbb{F}} = \begin{pmatrix} \sum_{i=0}^{M-1} \overline{H_0(\alpha_i)}\widehat{\mathbb{F}}_i \\ \vdots \\ \sum_{i=0}^{M-1} \overline{H_{N_1N_2-1}(\alpha_i)}\widehat{\mathbb{F}}_i \end{pmatrix} = \sum_{i=0}^{M-1} \widehat{\mathbb{F}}_i V_i.$$

Transpose Reduction

Similarly,

$$\begin{aligned}
\mathbb{H}^*\mathbb{H} &= \begin{pmatrix} \sum_{i=0}^{M-1} \overline{H_0(\alpha_i)} H_0(\alpha_i) & \cdots & \sum_{i=0}^{M-1} \overline{H_0(\alpha_i)} H_{N_1 N_2 - 1}(\alpha_i) \\ & \vdots & \\ \sum_{i=0}^{M-1} \overline{H_{N_1 N_2 - 1}(\alpha_i)} H_0(\alpha_i) & \cdots & \sum_{i=0}^{M-1} \overline{H_{N_1 N_2 - 1}(\alpha_i)} H_{N_1 N_2 - 1}(\alpha_i) \end{pmatrix} \\
&= \sum_{i=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_i)} H_0(\alpha_i) & \cdots & \overline{H_0(\alpha_i)} H_{N_1 N_2 - 1}(\alpha_i) \\ & \vdots & \\ \overline{H_{N_1 N_2 - 1}(\alpha_i)} H_0(\alpha_i) & \cdots & \overline{H_{N_1 N_2 - 1}(\alpha_i)} H_{N_1 N_2 - 1}(\alpha_i) \end{pmatrix} \\
&= \sum_{i=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_i)} \\ \overline{H_1(\alpha_i)} \\ \vdots \\ \overline{H_{N_1 N_2 - 1}(\alpha_i)} \end{pmatrix} \left(H_0(\alpha_i) \; H_1(\alpha_i) \; \cdots \; H_{N_1 N_2 - 1}(\alpha_i) \right) \\
&= \sum_{i=0}^{M-1} V_i V_i^* .
\end{aligned}$$

Transpose Reduction

To construct $A = \mathbb{H}^*\mathbb{H}$ and $b = \mathbb{H}^*\widehat{f}$:

1. Let $V_j = (H_0(\alpha_0), ..., H_{N_1 N_2 - 1}(\alpha_0))^*$
2. Set $A = V_j V_j^*$ and $b = \widehat{f}_0 V_j$
3. For $j = 0 : M - 1$
   - Set $V_j = (H_0(\alpha_j), ..., H_{N_1 N_2 - 1}(\alpha_j))^*$
   - $A = A + V_j V_j^*$
   - $b = b + \widehat{f}_j V_j$

Validation

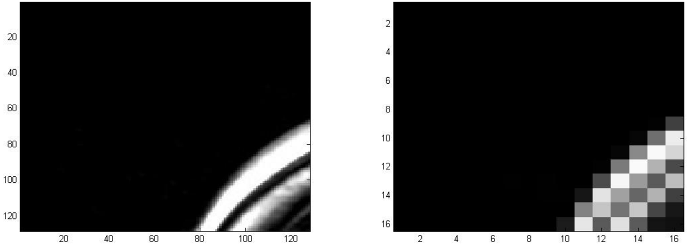


Figure: Left: High-resolution image, 128x128. Right: Ideal reconstruction, 16x16.

Timeline

- October 2015: Code the sampling routine to form the Fourier frame. **[Complete]**
- November 2015: Validation on small problems. **[Ongoing]**
- **December 2015: Code the transpose reduction algorithm [Complete] and begin testing.**
- January 2016: Code the conjugate gradient algorithm.
- February - March 2016: Error analysis/testing.
- April 2016: Finalize results.

Deliverables

- Synthetic data set
- Fourier frame sampling routine
- Downsampling routine
- Routine to generate spectral data
- Transpose reduction routine
- Conjugate gradient routine
- Final report and error analysis

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Thank you!

Questions?