MRI Reconstruction via Fourier Frames on Interleaving Spirals
Project Proposal

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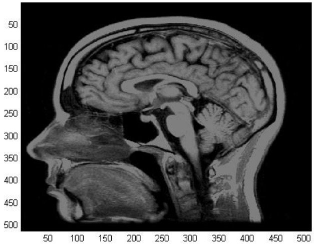
Background

MRI Reconstruction

- Common inversion problem: given frequency information, reconstruct image in the spatial domain
- Sampling on interleaving spirals make for fast data acquisition (Bourgeois et al)



“Carolyn’s MRI”, by ClintJCL (Flickr)

Paley-Wiener space

The *Paley-Wiener space* PW_E is defined as

$$PW_E = \{\varphi \in L^2(\widehat{\mathbb{R}}^d) : \text{supp } \varphi^\vee \subseteq E\},$$

where

- $\widehat{\mathbb{R}}^d$ is the spectral equivalent of $\mathbb{R}^d$
- $E \subseteq \mathbb{R}^d$ is compact
- $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\widehat{\mathbb{R}}^d)$ such that $\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \cdot \omega} dx$
- $\varphi^\vee = \mathcal{F}^{-1}(\varphi)$

Fourier Frame

A *frame* is a sequence $\{x_n : n \in \mathbb{Z}^d\} \subseteq H$, a separable Hilbert space, for which there exist $A, B > 0$ such that

$$\forall y \in H, \quad A||y||^2 \leq \sum_n |\langle y, x_n \rangle|^2 \leq B||y||^2.$$

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a sequence and let $E \subseteq \mathbb{R}^d$ be compact. The sequence $\{e_\lambda : \lambda \in \Lambda\}$, where $e_\lambda(x) = e^{-2\pi i x \cdot \lambda}$, defines a frame for PW_E if and only if there exist $0 < A \leq B < \infty$ such that

$$\forall \varphi \in PW_E, \quad A||\varphi||_{L^2(\widehat{\mathbb{R}}^d)} \leq \sum_{\lambda \in \Lambda} |\varphi(\lambda)|^2 \leq B||\varphi||_{L^2(\widehat{\mathbb{R}}^d)}.$$

We call such a sequence a *Fourier frame for PW_E* (Au-Yeung, Benedetto).

Beurling's Theorem (Beurling; Benedetto,Wu)

- Let $E = \overline{B(0,R)} \subset \mathbb{R}^d$.
- Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be uniformly discrete.
- Let $\text{dist}(\xi, \Lambda) = \inf_{\lambda \in \Lambda} \sqrt{\sum_{i=1}^{d} |\xi_i - \lambda_i|^2}$ denote the Euclidean distance between the point $\xi \in \widehat{\mathbb{R}}^d$ and the set Λ.

Define

$$\rho = \rho(\Lambda) = \sup_{\xi \in \widehat{\mathbb{R}}^d} \text{dist}(\xi, \Lambda).$$

If $R\rho < \frac{1}{4}$, then Λ is a Fourier frame for $PW_{\overline{B(0,R)}} \subseteq L^2(\widehat{\mathbb{R}}^d)$.

Every finite energy signal $f \in L^2(E)$ can thus be represented as

$$f(x) = \sum_{\lambda \in \Lambda} a_\lambda(f) e_\lambda \mathbb{1}_E$$

Fourier Frame on Interleaving Spirals

Let $\{A_k : k = 0, 1, ..., M-1\} \subseteq \widehat{\mathbb{R}}^d$ denote a finite set of interleaving Archimedean spirals of the form

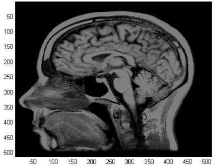
$$A_k = \{c\theta e^{2\pi i(\theta-(k/M))} : \theta \geq 0\}.$$

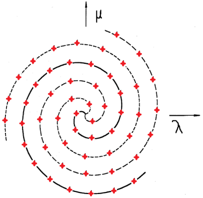
1. Choose $\delta > 0$ such that $R\rho < 1/4$.
2. For each k, choose a uniformly discrete set Λ_k of points along A_k where the curve distance between consecutive points is less than 2δ, beginning within 2δ of the origin.

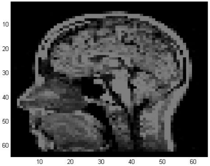
The set $\Lambda_R = \cup_{k=0}^{M-1} \Lambda_k \subseteq B = \cup_{k=1}^{M-1} A_k$ defines a Fourier frame for $PW_{B(0,R)}$ (Benedetto,Wu).

Problem Overview

The goal of this project is to use nonuniform sampling on interleaving spirals to define a Fourier frame in $\widehat{\mathbb{R}}^d$, from which we can reconstruct a low-resolution MRI image.







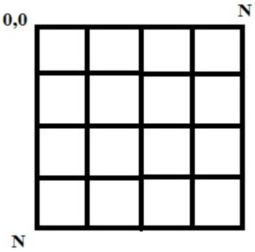
Images courtesy of U.S. Patent US5485086 A and “Carolyn’s MRI”, by ClintJCL (Flickr)

$$f(p) = f(x,y) = \sum_{b_j=0}^{N^2-1} f_{b_j} \mathbb{1}_{\Box_{b_j}}(x,y)$$

$$\hat{f}(\alpha) = \hat{f}(\lambda,\mu) = \sum_{b_j=0}^{N^2-1} f_{b_j} H_{b_j}(\lambda,\mu)$$

where

$$\begin{aligned} H_{b_j}(\alpha) &= H_{b_j}(\lambda,\mu) \\ &= \widehat{\mathbb{1}}_{\Box_{b_j}}(\lambda,\mu) \end{aligned}$$



Computational Approach

Choose $M \geq N^2$ points $\alpha_i = (\lambda_i, \mu_i)$ on the interleaving spirals to get

$$\widehat{f}(\alpha_i) = \sum_{j=0}^{N^2-1} f_{b_j} H_{b_j}(\alpha_i). \tag{1}$$

Let

$$\widehat{\mathbb{F}} = [\widehat{f}(\alpha_0)\ \widehat{f}(\alpha_1)\ \ldots\ \widehat{f}(\alpha_{M-1})]^{\mathrm{T}}$$

and

$$\mathbb{F} = [f_{b_0}\ f_{b_1}\ \ldots\ f_{b_{N^2-1}}]^{\mathrm{T}}.$$

Let $\mathbb{H} = [H_{b_j}(\alpha_i)]_{i,j}$, and (1) becomes

$$\widehat{\mathbb{F}} = \mathbb{H}\mathbb{F}. \tag{2}$$

Implementation

We will solve the least-squares problem

$$\mathbb{F} = (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{\mathbb{F}}, \tag{3}$$

where

- $\mathbb{H}$ is the Bessel map $\ell^2(\{0, 1, ..., N^2 - 1\}) \to \ell^2(\{0, 1, ..., M - 1\})$
- $\mathbb{H}^*$ is its adjoint
- $\mathbb{H}^*\mathbb{H}$ is the frame operator

We consider the following algorithms:

- Transpose reduction (direct approach with efficient storage)
- Conjugate gradient algorithm

Transpose Reduction

Let $A = \mathbb{H}^*\mathbb{H}$ and $b = \mathbb{H}^*\widehat{f}$. Define $V_j = (H_0(\alpha_j), ..., H_{N^2-1}(\alpha_j))^*$ s.t.

$$\mathbb{H} = \begin{pmatrix} H_0(\alpha_0) & \cdots & H_{N^2-1}(\alpha_0) \\ H_0(\alpha_1) & \cdots & H_{N^2-1}(\alpha_1) \\ \vdots & \vdots & \vdots \\ H_0(\alpha_{M-1}) & \cdots & H_{N^2-1}(\alpha_{M-1}) \end{pmatrix} = \begin{pmatrix} V_0^{\mathrm{T}} \\ V_1^{\mathrm{T}} \\ \vdots \\ V_{M-1}^{\mathrm{T}} \end{pmatrix}.$$

Note that

$$\begin{aligned} A = \mathbb{H}^*\mathbb{H} &= \sum_{k=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_k)}H_0(\alpha_k) & \cdots & \overline{H_0(\alpha_k)}H_{N^2-1}(\alpha_k) \\ & \vdots & \\ \overline{H_{N^2-1}(\alpha_k)}H_0(\alpha_k) & \cdots & \overline{H_{N^2-1}(\alpha_k)}H_{N^2-1}(\alpha_k) \end{pmatrix} \\ &= \sum_{k=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_k)} \\ \overline{H_1(\alpha_k)} \\ \vdots \\ \overline{H_{N^2-1}(\alpha_k)} \end{pmatrix} (H_0(\alpha_k)\ H_1(\alpha_k)\ \cdots\ H_{N^2-1}(\alpha_k)) \\ &= \sum_{k=0}^{M-1} V_k V_k^*. \end{aligned}$$

Transpose Reduction

$$b = \mathbb{H}^* \widehat{f} = \begin{pmatrix} \sum_{k=0}^{M-1} \overline{H_0(\alpha_k)} \widehat{f}_k \\ \vdots \\ \sum_{k=0}^{M-1} \overline{H_{N^2-1}(\alpha_k)} \widehat{f}_k \end{pmatrix} = \sum_{k=0}^{M-1} \widehat{f}_k V_k.$$

To construct $A = \mathbb{H}^*\mathbb{H}$ and $b = \mathbb{H}^*\widehat{f}$:

1. Let $V_j = (H_0(\alpha_0), ..., H_{N^2-1}(\alpha_0))^*$
2. Set $A = V_j V_j^*$ and $b = \widehat{f}_0 V_j$
3. For $j = 0 : M - 1$
   - Set $V_j = (H_0(\alpha_j), ..., H_{N^2-1}(\alpha_j))^*$
   - $A = A + V_j V_j^*$
   - $b = b + \widehat{f}_j V_j$

Factor of N^2/M less memory than direct approach with naive storage.

Conjugate Gradient Method

Let $A = \mathbb{H}^*\mathbb{H}$ and $b = \mathbb{H}^*\widehat{f}$. To solve $Af = b$ for symmetric, positive definite A:

1. Choose f_0. Let $r_0 = b - Af_0$. Set $p_0 = r_0$.
2. for $n = 1$ until convergence
   - $\gamma = (r_n^{\mathrm{T}} r_n)/((Ap_n)^{\mathrm{T}} p_n)$
   - $f_{n+1} = f_n + \gamma p_n$
   - $r_{n+1} = r_n - \gamma A p_n$
   - if $\mathrm{norm}(r_{n+1}) < \mathrm{tol}$, break
   - $\beta_n = (r_{n+1}^{\mathrm{T}} r_{n+1})/(r_n^{\mathrm{T}} r_n)$
   - $p_{n+1} = r_{n+1} + \beta_n p_n$

Generally has linear convergence, but the speed of convergence depends on the condition number of A.

Validation and Testing

Validation

- Small problems (on the order of 64 $\times$ 64 pixels) can be solved directly.
- CG algorithm should follow the same convergence trajectory as Matlab's version.

Testing will primarily consist of error analysis.

Error measures:

- Signal-to-noise ratio (SNR)
- Structural Similarity measure (SSIM) (Wang et al)

Software: Matlab. Hardware: Acer Aspire V5 (6GB RAM)

Timeline

- October 2015: Code the sampling routine to form the Fourier frame.
- November 2015: Proof of concept on small problems.
- December 2015: Code the transpose reduction algorithm and begin testing.
- January 2016: Code the conjugate gradient algorithm.
- February - March 2016: Error analysis/testing. Explore how much frequency information we need to adequately recover f. Explore condition number of $\mathbb{H}^*\mathbb{H}$ and how it affects the reconstruction.
- April 2016: Finalize results.

Deliverables

- Synthetic data set
- Fourier frame sampling routine
- Transpose reduction routine
- Conjugate gradient routine
- Final report and error analysis

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Thank you!

Questions?