Solving the Stochastic Steady-state Diffusion Problem Using Multigrid

Tengfei Su
Applied Mathematics and Scientific Computing

Advisor: Howard Elman
Department of Computer Science

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Outline

Project goal

The stochastic steady-state diffusion equation

$$\begin{cases} -\nabla \cdot (c(x,\omega)\nabla u(x,\omega)) = f(x) & \text{in } D \times \Omega \\ u(x,\omega) = 0 & \text{on } \partial D \times \Omega \end{cases}$$

with stochastic coefficient $c(x,\omega) : D \times \Omega \to \mathbb{R}$.

- Approach: stochastic finite element method (SFEM) [Ghanem & Spanos, 2003]
- Solver: multigrid [Elman & Furnival, 2007]

Stochastic FEM

Karhunen-Loève expansion [Powell & Elman, 2009]

$$c(x,\omega) = c_0(x) + \sum_{k=1}^{m} \sqrt{\lambda_k} c_k(x) \xi_k(\omega).$$

Weak form

$$\int_\Gamma \rho(\xi) \int_D c(x,\xi) \nabla u(x,\xi) \cdot \nabla v(x,\xi) \mathrm{d}x \mathrm{d}\xi$$
$$= \int_\Gamma \rho(\xi) \int_D f(x) v(x,\xi) \mathrm{d}x \mathrm{d}\xi,$$

where $\rho(\xi)$ is the joint density function, Γ is the joint image of $\{\xi_k\}_{k=1}^m$.

Stochastic FEM

Finite-dimensional subspace

$$V^h = S \otimes T = \text{span}\{\phi(x)\psi(\xi), \phi \in S, \psi \in T\}$$

with basis functions

- $\phi(x)$: piecewise linear/bilinear basis function
- $\psi(\xi)$: m-dimensional orthogonal polynomials of order p [Xiu & Karniadakis, 2003].

SFEM solution

$$u_{hp}(x, \xi) = \sum_{j=1}^{N} \sum_{s=1}^{M} u_{js} \phi_j(x) \psi_s(\xi).$$

Galerkin system

Find $\mathbf{u} \in \mathbb{R}^{MN}$, such that

$$A\mathbf{u} = \mathbf{f}.$$

Using tensor product notation,

$$A = G_0 \otimes K_0 + \sum_{k=1}^{m} G_k \otimes K_k,$$

where

$$G_0(r,s) = \int_\Gamma \psi_r(\xi)\psi_s(\xi)\rho(\xi)\mathrm{d}\xi$$

$$G_k(r,s) = \int_\Gamma \xi_k\psi_r(\xi)\psi_s(\xi)\rho(\xi)\mathrm{d}\xi, \quad r,s = 1,\ldots,M$$

$$K_0(i,j) = \int_D c_0(x)\nabla\phi_i(x)\nabla\phi_j(x)\mathrm{d}x$$

$$K_k(i,j) = \int_D \sqrt{\lambda_k}c_k(x)\nabla\phi_i(x)\nabla\phi_j(x)\mathrm{d}x, \quad i,j = 1,\ldots,N$$

Galerkin system

Galerkin solution

$$\mathbf{u} = [u_{11}, u_{21}, \ldots, u_{N1}, \ldots, u_{1M}, u_{2M}, \ldots, u_{NM}]^T$$

For the right-hand side,

$$\mathbf{f} = g_0 \otimes f_0,$$

where

$$g_0(r) = \int_\Gamma \psi_r(\xi)\rho(\xi)\mathrm{d}\xi, \quad r = 1, \ldots, M$$

$$f_0(i) = \int_D f(x)\phi_i(x)\mathrm{d}x, \quad i = 1, \ldots, N$$

Progress: SFEM

Using IFISS and SIFISS [Silvester et al] to generate the Galerkin system:

- G, K matrices
- rhs vector $\mathbf{f}$
- mesh data
- other input parameters

Main task: writing a multigrid solver for

$$(G_0 \otimes K_0 + \sum_{k=1}^{m} G_k \otimes K_k)\mathbf{u} = \mathbf{f}.$$

Two-grid Correction Scheme

- **Algorithm**

  Choose initial guess $\mathbf{u}^{(0)}$
  for $i = 0$ until convergence
  for k steps
  $\mathbf{u}^{(i)} \leftarrow \mathbf{u}^{(i)} + Q^{-1}(\mathbf{f} - A\mathbf{u}^{(i)})$
  end
  $\bar{\mathbf{r}} = \mathscr{R}(\mathbf{f} - A\mathbf{u}^{(i)})$
  solve $\bar{A}\bar{\mathbf{e}} = \bar{\mathbf{r}}$
  $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \mathscr{P}\bar{\mathbf{e}}$
  for k steps
  $\mathbf{u}^{(i+1)} \leftarrow \mathbf{u}^{(i+1)} + Q^{-1}(\mathbf{f} - A\mathbf{u}^{(i+1)})$
  end
  end

Two-grid Correction Scheme

- **Algorithm**

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  for k steps

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  $\bar{\mathbf{r}} = \mathscr{R}(\mathbf{f} - A\mathbf{u}^{(i)})$

  solve $\bar{A}\bar{\mathbf{e}} = \bar{\mathbf{r}}$

  $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \mathscr{P}\bar{\mathbf{e}}$

  for k steps

  $\mathbf{u}^{(i+1)} \leftarrow \mathbf{u}^{(i+1)} + Q^{-1}(\mathbf{f} - A\mathbf{u}^{(i+1)})$

  end

  end

Prolongation Operator

- Fine grid space

$$V^h = T^p \otimes S^h, \dim(V^h) = M \times N_h$$

coarse grid space

$$V^{2h} = T^p \otimes S^{2h}, \dim(V^{2h}) = M \times N_{2h}$$

- Any basis function $\phi_j^{2h} \in S^{2h}$ can be written as

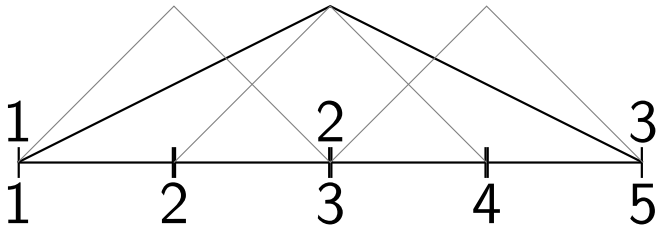
$$\phi_j^{2h} = \sum_{i=1}^{N_h} p_{ij} \phi_i^h, \; j = 1, \ldots, N_{2h}$$

- Prolongation operator

$$\mathscr{P} = I \otimes P, \text{ with } P_{ij} = p_{ij}$$

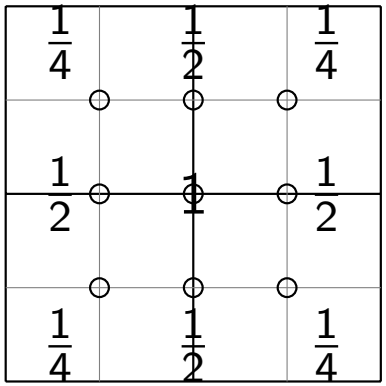
Prolongation Operator

- In 1D



$$\phi_2^{2h} = \frac{1}{2}\phi_2^h + \phi_3^h + \frac{1}{2}\phi_4^h$$

- In 2D



Prolongation Operator

The matrix P $(N_h \times N_{2h})$

0.2500	0	0	0	0	0	0	0	0
0.5000	0	0	0	0	0	0	0	0
0.2500	0.2500	0	0	0	0	0	0	0
0	0.5000	0	0	0	0	0	0	0
0	0.2500	0.2500	0	0	0	0	0	0
0	0	0.5000	0	0	0	0	0	0
0	0	0.2500	0	0	0	0	0	0
0.5000	0	0	0	0	0	0	0	0
1.0000	0	0	0	0	0	0	0	0
0.5000	0.5000	0	0	0	0	0	0	0
0	1.0000	0	0	0	0	0	0	0
0	0.5000	0.5000	0	0	0	0	0	0
0	0	1.0000	0	0	0	0	0	0
0	0	0.5000	0	0	0	0	0	0
0.2500	0	0	0.2500	0	0	0	0	0
0.5000	0	0	0.5000	0	0	0	0	0
0.2500	0.2500	0	0.2500	0.2500	0	0	0	0
0	0.5000	0	0	0.5000	0	0	0	0
0	0.2500	0.2500	0	0.2500	0.2500	0	0	0
0	0	0.5000	0	0	0.5000	0	0	0
0	0	0.2500	0	0	0.2500	0	0	0
0	0	0	0.5000	0	0	0	0	0
0	0	0	1.0000	0	0	0	0	0
0	0	0	0.5000	0.5000	0	0	0	0
0	0	0	0	1.0000	0	0	0	0
0	0	0	0	0.5000	0.5000	0	0	0
0	0	0	0	0	1.0000	0	0	0
0	0	0	0	0	0.5000	0	0	0

Construction of $\bar{A}$

- Restriction operator

$$\mathscr{R} = I \otimes P^T$$

- Construction of $\bar{A}$

$$\bar{A} = \mathscr{R} A \mathscr{P} = G_0 \otimes (P^T K_0 P) + \sum_{k=1}^{m} G_k \otimes (P^T K_k P)$$

- For the diffusion problem [Briggs et al, 2000],

$$P^T K_k^h P = K_k^{2h},\ k = 0, \ldots, m$$

i.e. we can set up K matrices directly on the coarse grid.

Smoother

- Damped Jacobi

$$A = D - (-L - U),\ Q = \frac{1}{\omega} D$$

$$\mathbf{u}^{(i)} \leftarrow \mathbf{u}^{(i)} + \omega D^{-1}(\mathbf{f} - A\mathbf{u}^{(i)})$$

- For A

$$\begin{aligned}\mathrm{diag}(A) &= \mathrm{diag}(G_0 \otimes K_0 + \sum_{k=1}^{m} G_k \otimes K_k) \\ &= \mathrm{diag}(G_0 \otimes K_0)\end{aligned}$$

Matrix Vector Product

- Galerkin solution

$$\mathbf{u} = [u_{11}, \ldots, u_{N1}, \ldots, u_{1M}, \ldots, u_{NM}]^{\mathsf{T}}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1M} \\ u_{21} & u_{22} & \cdots & u_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & \cdots & u_{NM} \end{pmatrix}$$

- Matrix $A = G_0 \otimes K_0 + \sum_{k=1}^{m} G_k \otimes K_k$

$$\Rightarrow A\mathbf{u} = \mathrm{vec}(K_0 U G_0 + \sum_{k=1}^{m} K_k U G_k)$$

- Prolongation $(I \otimes P)\bar{\mathbf{e}}$, restriction $(I \otimes R)\mathbf{r}$, smoother $D^{-1}\mathbf{r}$

One Multigrid Iteration: V-cycle


```
function x=stoch_mg_iter(Ks,G,x0,f,smooth_data,level,npre
    ,npost)

K=Ks(level).matrix; P{1}=Ks(level).prolong;
R{1}=P{1}'; I{1}=speye(size(G{1}));

if level==2
   dimk=length(K);
   A=kron(G{1},K{1});
   for dim=2:dimk
       A=A+kron(G{dim},K{dim});
   end
   x=A\f;
else
   x=stoch_mg_pre(K,G,x0,f,npre,smooth_data,level);
   r=f-stoch_matvec(x,G,K);
   rc=stoch_matvec(r,I,R);
   cc=stoch_mg_iter(Ks,G,zeros(size(rc)),rc,smooth_data,
       level-1,npre,npost);
   x=x+stoch_matvec(cc,I,P);
   x=stoch_mg_post(K,G,x,f,npost,smooth_data,level);
end
```

Multigrid Solver

- Construct K, P, Q^{-1} on each grid level
- $\mathbf{u}^{(0)} = \mathbf{0}$, $r0 = \text{norm}(\mathbf{f} - A\mathbf{u}^{(0)})$, $i = 0$
- while $r > tol * r0$ & $i <= maxit$
  - execute one multigrid iteration for $A\mathbf{u} = \mathbf{f}$
    $\mathbf{u}^{(i+1)} = \text{stoch\_mg\_iter}(A, \mathbf{u}^{(i)}, \mathbf{f}, \ldots)$
    $r = \text{norm}(\mathbf{f} - A\mathbf{u}^{(i+1)})$
    $i = i + 1$

  or

  - execute one multigrid iteration for $A\mathbf{e} = \mathbf{r}$
    $\mathbf{e}^{(i+1)} = \text{stoch\_mg\_iter}(A, \mathbf{0}, \mathbf{f} - A\mathbf{u}^{(i)}, \ldots)$
    $\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \mathbf{e}^{(i+1)}$
    $r = \text{norm}(\mathbf{f} - A\mathbf{u}^{(i+1)})$
    $i = i + 1$

  end

Model Problem

- $D = (-1, 1)^2, f = 1$. Covariance function

$$r(x, y) = \sigma^2 \exp(-\frac{1}{b_1}|x_1 - y_1| - \frac{1}{b_2}|x_2 - y_2|)$$

- KL expansion with $c_0(x) = 1$

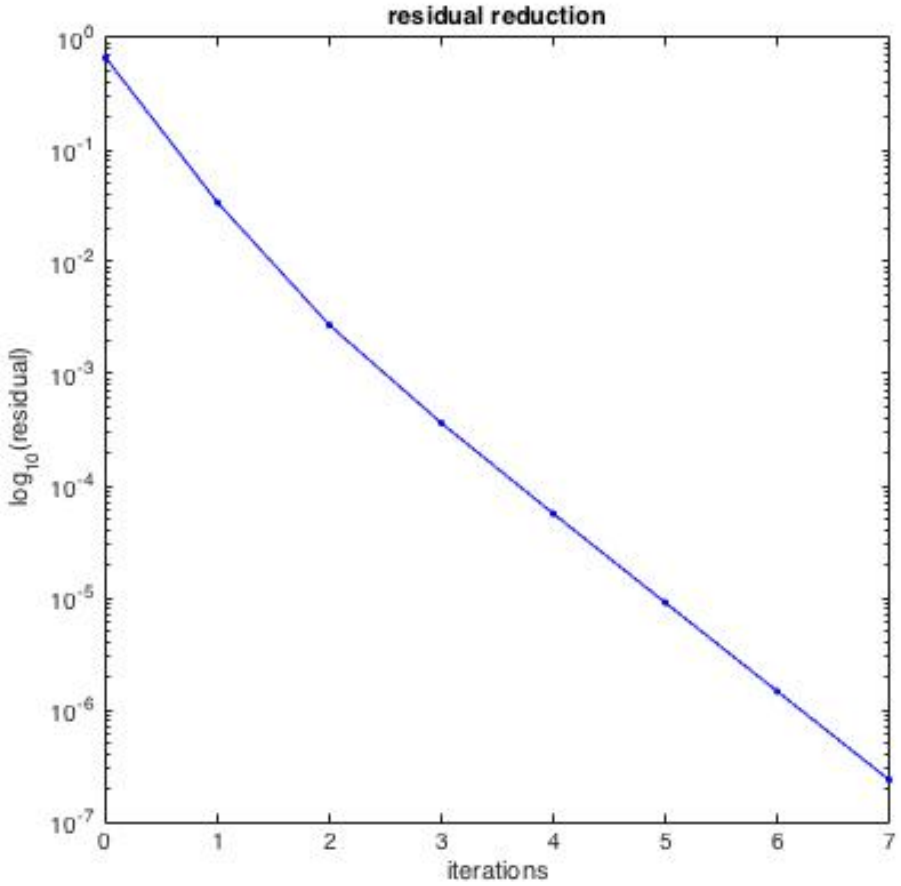
$$c(x, \omega) = c_0(x) + \sqrt{3}\sigma \sum_{k=1}^{m} \sqrt{\lambda_k} c_k(x) \xi_k(\omega)$$

- Uniform distribution (assuming independence)

$$\xi_k \sim \mathrm{U}(-1, 1),\ \rho(\xi) = \frac{1}{2^m}$$

Model Problem

- $\sigma = 0.3, b_1 = b_2 = 2$
- $m = 3, p = 3, tol = 10^{-6}$



Validation: Convergence Performance

- Independent of h

$m = 3, p = 3$

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
n	6	7	7	7	7	8

$m = 5, p = 3$

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
n	7	8	8	8	8	8

$m = 3, p = 5$

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
n	7	8	8	8	8	9

Validation: Convergence Performance

- Independent of m, p (take $h = 2^{-3}$)

$(m = 2)$ p	1	2	3	4	5	6
n	6	6	7	7	7	7

$(m = 3)$ p	1	2	3	4	5	6
n	6	6	7	7	8	8

$(p = 2)$ m	1	2	3	4	5	6
n	6	6	6	7	7	7

$(p = 3)$ m	1	2	3	4	5	6
n	6	7	7	7	8	8

Validation: Monte Carlo

- Galerkin solution

$$u_{hp}(x,\xi) = \sum_{j=1}^{N}\sum_{s=1}^{M} u_{js}\phi_j(x)\psi_s(\xi)$$

- Orthogonality $\mathbb{E}[\psi_i(\xi)\psi_j(\xi)] = \delta_{ij}$

$$\psi_1(\xi) = 1 \Rightarrow \mathbb{E}[\psi_i(\xi)] = \delta_{i1}$$

$$\mathbb{E}[u_{hp}(x,\xi)] = \sum_{j=1}^{N} u_{j1} \Rightarrow \mathbb{E}[u_{hp}(x_j,\xi)] = u_{j1}$$

- Variance $\mathbb{V}[u_{hp}(x_j,\xi)] = \sum_{s=2}^{M} u_{js}^2$

Validation: Monte Carlo

- Monte Carlo method [Lord et al, 2014]

$$\xi_k \sim \mathrm{U}(-1,1) \Rightarrow \xi = 2 * \mathrm{rand}(m,1) - 1$$

$$(K_0 + \sum_{k=1}^{m} \xi_k K_k)\mathbf{u} = \mathbf{f}$$

- Compute mean and variance

$$\mathbb{E}[u_{MC}] = \frac{1}{n}\sum_{i=1}^{n} u_{MC}^i$$

$$\mathbb{V}[u_{MC}] = \frac{1}{n-1}\sum_{i=1}^{n}(u_{MC}^i - \mathbb{E}[u_{MC}])^2$$

$$= \frac{1}{n-1}(\sum_{i=1}^{n}(u_{MC}^i)^2 - n\mathbb{E}[u_{MC}]))$$

Schedule

- Generate Galerkin system from IFISS/S-IFISS ✓
- Write the multigrid solver and implement for model problem
  - Uniform distributions ✓
  - Normal distributions ✗
- Validation
  - Convergence performance ✓
  - Comparison with Monte Carlo (in progress)
- Mid-year presentation ✓

Schedule

- Seek low-rank approximate solutions for [Kressner & Tobler, 2011]

$$(G_0 \otimes K_0 + \sum_{i=1}^{m} G_i \otimes K_i)\mathbf{u} = \mathbf{f}$$

- Write

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1M} \\ u_{21} & u_{22} & \cdots & u_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & \cdots & u_{NM} \end{pmatrix}$$

$$K_0 U G_0 + \sum_{i=1}^{m} K_i U G_i = F$$

- $U \approx U_k = V_k W_k^{\mathsf{T}}$, $V_k \in \mathbb{R}^{N \times k}$, $W_k \in \mathbb{R}^{M \times k}$, $k \ll N, M$

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The End

- Thank you!
- Questions?