

CBMS Lectures plan: Families of Riemann surfaces and Weil-Petersson geometry

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Our goal is to use the range of topics of investigation of Weil-Petersson geometry as an opportunity to introduce general concepts. We will describe recurring methods and arguments, as well as a selection of results. Our approach will not be properly historical nor developmental. Also, we will not be able to include a number of important topics in the lectures. We hope to provide participants with tools towards understanding the literature and current work.

Preliminaries (1 lecture)

- Identification spaces, Riemann surfaces and line bundles
- Hyperbolic geometry & Uniformization
- Injectivity radius, thick-thin decompositions and the enhanced collar lemma
- Mumford compactness & Gromov-Hausdorff compactness of thick regions
- Right hexagons and Fenchel-Nielsen (FN) coordinates
- The family $\{zw = t\} \rightarrow \{t\}$: complex geometry, regular n -differentials, family hyperbolic metrics, and collars tending to cusps

Teichmüller space and explicit deformation of a horizontal strip (1 lecture)

- Definition of Teichmüller space
- Deformation of a discrete translation group in the approach of: group cocycles, sliding plates (Čech cocycles), deformation vector fields, quasi conformal maps and Beltrami differentials
- A formula for the n^{th} variation of translation length
- Stretching, twisting and twist-length duality in the WP pre Kähler form

Geodesic-length functions, FN twists and WP symplectic geometry (1 lecture)

- Convergence of deck group sums; folding & unfolding
- Geodesic-length functions ℓ_α and FN twists for the deck group automorphic setting; twist-length duality
- Calculus of twist derivatives & cross ratios
- Riera's gradient-pairing formula for $\langle \text{grad } \ell_\alpha, \text{grad } \ell_\beta \rangle$
- Deformation vector fields and Eichler type integrals
- Positivity of the Hessian $\text{Hess } \ell_\alpha$ of geodesic-length functions
- Nielsen Realization
- $\omega = d\ell \wedge d\tau$ by symmetry considerations

Higher-order Masur expansions and the geometry of the augmented Teichmüller space (3 lectures)

- Expansion of the WP metric, Levi-Civita connection and Riemann tensor in terms of geodesic-length gradients
- Yamada normal form and twist angles
- Complex of curves $C(F)$
- Definition of the augmented Teichmüller space $\overline{\mathcal{T}}$ - a partial compactification
- WP completion and non refraction of geodesics on $\overline{\mathcal{T}}$
- $\overline{\mathcal{T}}$ is $CAT(0)$
- The Bers bounded partition theorem
- Gradients of geodesic-length functions & tangents of terminating geodesics
- Brock's approximation & $\overline{\mathcal{T}}$ as an infinite polyhedron
- Isometries of the WP polyhedron and the Masur-Wolf theorem
- The pants graph $\mathcal{P}(F)$ and Brock's quasi isometry of $\overline{\mathcal{T}}$
- Bers regions and mapping class group (MCG) rough fundamental domains
- WP geodesics & ending measured geodesic laminations
- The structure of WP Alexandrov tangent cones of $\overline{\mathcal{T}}$

- (formal) Coxeter group of curves $CG(F)$ & development \mathcal{D} of $\overline{\mathcal{T}}$ by $CG(F)$
- Yamada's theorem: \mathcal{D} is $CAT(0)$ and FR (finite rank)
- Daskalopoulos-Wentworth rigidity

Variation of hyperbolic metrics and WP related curvatures (1 lecture)

- Hyperbolic covariant differentiation
- The Prescribed Curvature Equation
- Maximum principle and interior Schauder estimates for the prescribed curvature equation
- Variation of the Laplace-Beltrami operator & Gauss curvature
- Second variation of hyperbolic area and variation of harmonic projection
- WP Riemann tensor
- General method for variation of hyperbolic metrics: estimates & formulas

Sums of $e^{-2 \text{distance}}$ and collar expansions (1 lecture)

- Expansions
- Ubiquitous e^{-2d} formulas
- Collar Principle
- Mean value & the exterior sum estimate
- Expanding & bounding: $\Theta_\alpha, \mathbb{P}_\alpha, \langle \text{grad } \ell_\alpha, \text{grad } \ell_\beta \rangle$ and $(D - s(s - 1))^{-1}$
- Role in curvature applications of Takhtajan-Zograf, Liu-Sun-Yau and Freixas i Montplet

Mirzakhani's integration recursion & simple prime geodesic theorem (2 lectures)

- McShane's identity generalized
- The integration recursion and formulas
- Measured laminations \mathcal{ML} , Thurston's volume & MCG ergodicity
- Train-track volume in \mathcal{ML} & the counting of multi curves (lattice points)
- The chorus: the simple prime geodesic theorem